Elementary Introduction to the Theory of Automorphic forms
Lecture8+HW3

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There are four problems of HW. They are marked by !!!! send solutions to alevin57@gmail.com until 21.04

Another θs

I present more theta functions, which I will use later.

$$\theta_{00}(\xi;\tau) = \sum_{k\in\mathbb{Z}} \exp\left(2\pi i\left(\frac{k^2}{2}\tau + k\xi\right)\right).$$

Easily $\theta_{00}(\xi + \frac{\tau}{2} + \frac{1}{2};\tau) = \exp\left(-2\pi i\left(\frac{1}{8}\tau + \frac{1}{2}(\xi + \frac{1}{2})\right)\right)\theta_{11}(\xi;\tau)$.
Note that the quasiperiodicity of θ can be reinterpret in the following way: For integer m, n put

$$T_{m,n}(f(\xi;\tau)) = \exp\left(2\pi i\left(\frac{m^2}{2}\tau + m(\xi + n)\right)\right) f(\xi + m\tau + n;\tau)$$

. Then $T_{m,n}(\theta_{11}(\xi;\tau)) = (-1)^{(m+n}\theta_{11}(\xi;\tau)$ and $T_{m,n}(\theta_{00}(\xi;\tau)) = \theta_{00}(\xi;\tau)$.
One can interpolate the operator T_{**} to the rational indices, then $\theta_{11}(\xi;\tau) = T_{\frac{1}{2},\frac{1}{2}}\theta_{00}(\xi;\tau)$. For any pair α, β, α, β are either 0 or 1 put $\theta_{\alpha\beta}(\xi;\tau) = T_{\frac{\alpha}{2},\frac{\beta}{2}}\theta_{00}(\xi;\tau)$. For more details see Mumford.

The θ-functions and the Elliptic Functions.

Temporarily for short hands we omit indices 11 and variable τ in notations, so $\theta(\xi)$ denotes $\theta_{11}(\xi;\tau)$, $\theta'(0)$ denotes $\partial\theta_{11}(\xi;\tau)/\partial\xi|_{\xi=0}$ and $\wp(\xi)$ denotes $\wp(\xi;\tau)$.

Lemma

$$\wp(\xi) - \wp(\eta) = \frac{\theta'(0)^2\theta(\xi+\eta)\theta(\eta-\xi)}{\theta(\xi)^2\theta(\eta)^2}.$$

Proof HW1!!! the idea is the following:
first check for $\eta \neq m\tau + n$, $m, n \in \mathbb{Z}$ that the rhs is elliptic in ξ;
second check for $\eta \neq m\tau + n$, $m, n \in \mathbb{Z}$ that the ratio of lhs and the rhs is regular everywhere (for this use information about zeros of the θ-function), hence does not depend in ξ;
third calculate this constant by tending to 0.

Lemma

$$\wp'(\xi) = -2\frac{\theta'(0)^3\theta(\xi+\frac{1}{2})\theta(\xi+\frac{\tau}{2})\theta(\xi-\frac{1}{2}-\frac{\tau}{2})}{\theta(\xi)^3\theta(\frac{1}{2})\theta(\frac{\tau}{2})\theta(-\frac{1}{2}-\frac{\tau}{2})}.$$

Proof HW2 !!!
The factor $\frac{\theta'(0)^3}{\theta(\frac{1}{2})\theta(\frac{\tau}{2})\theta(-\frac{1}{2}-\frac{\tau}{2})}$ can be recalculated.
The expressions $\wp(\frac{1}{2})$, $\wp(\frac{\tau}{2})$ and $\wp(-\frac{1}{2}-\frac{\tau}{2})$ are roots of the cubic equation $4x^3 - 60e_4x - 140e_6 = 0$ The discriminant of this equation equals to

$$\left(\left(\wp(\frac{1}{2})-\wp(\frac{\tau}{2})\right)\left(\wp(-\frac{1}{2}-\frac{\tau}{2})-\wp(\frac{1}{2})\right)\left(\wp(\frac{\tau}{2})-\wp(-\frac{1}{2}-\frac{\tau}{2})\right)\right)^2 =$$

$$\left(\frac{\theta'(0)^2\theta(\frac{1}{2}+\frac{\tau}{2})\theta(\frac{1}{2}-\frac{\tau}{2})}{\theta(\frac{1}{2})^2\theta(\frac{\tau}{2})^2}\right)^2 \times \cdots$$

and is a modular form as polynomial in e_4 and e_6.

Lemma

$$\left(\frac{\theta(\frac{1}{2}+\frac{\tau}{2})\theta(\frac{1}{2}-\frac{\tau}{2})}{\theta(\frac{1}{2})^2\theta(\frac{\tau}{2})^2}\right)^2 \times \cdots = \left(\theta_{00}(0)\theta_{01}(0)^2\theta_{10}(0)^2\right)^{-2}$$

Proof HW 3 !!!!!

Lemma

$\theta'_{11}(0) = -\pi\theta_{00}(0)\theta_{01}(0)\theta_{10}(0)$

Proof HW 4!!! the idea is the following

first, according to previous speculations, $\theta'_{11}(0)^{12}\left({}_{00}(0)^2\theta_{01}(0)^2\theta_{10}(0)^2\right)^{-2}$ is a modular form of the weight 12 and it vanishes at $\tau = i\infty$;

second, according to previous lecture, $\theta'_{11}(0)^8$ also satisfies these conditions:

third, from consideration of zeroes of modular forms conclude that these forms are proportional;

forth, we have proved that $\theta'_{11}(0) = K\theta_{00}(0)\theta_{01}(0)\theta_{10}(0)$ for some constant K. This constant can be evaluated by $\tau \to i\infty$

Infinite products

Lemma

$$\theta_{11}(\xi;\tau) = i\exp(2\pi i\frac{\tau}{8})(\exp(2\pi i\frac{\xi}{2}) - \exp(-2\pi i\frac{\xi}{2}))\times$$

$$\times\prod_{j=1}^{\infty}(1-\exp(2\pi i(\xi+j\tau))(1-\exp(2\pi i(-\xi+j\tau))(1-\exp(2\pi i(\xi+j\tau))$$

and analogous for other $\theta_{\alpha\beta}$.

Proof HW 4 !!!! Idea of the prove:
first, as above, prove that the ratio of the lhs and the rhs does no depends in ξ
second, the substitution
$\theta_{11}(\xi;\tau) = Ki\exp(2\pi i\frac{\tau}{8})(\exp(2\pi i\frac{\xi}{2}) - \exp(-2\pi i\frac{\xi}{2}))\cdots$ to
$\theta'_{11}(0) = -\pi\theta_{00}(0)^2\theta_{01}(0)\theta_{10}(0)$ get $K^2 = 1$, for this use equality
$\prod_{j=1}^{\infty}(1-\exp((2j-1)a)(1+\exp((2j-1)a)(1+\exp(2ja)) = 1$
The last can be proved by multiplication by $\prod_{j=1}^{\infty}(1-\exp(2ja))$