# Elementary Introduction to the Theory of Automorphic forms
# Lecture9

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# Modular Functions

## Definition

*A holomorphic function $f$ on the upper half-plane is called modular function if*
*a)it is invariant with respect to the fraction-linear action of $\mathrm{SL}_2(\mathbb{Z})$.*
*b)it has finite order pole at $0$ as function in $q = \exp(2\pi i\tau)$:*
$f(\tau) = \sum_{k=-N}^{\infty} a_k \exp(2\pi i k\tau)$.

*Example.* The ratio

$$j = \frac{(60e_4)^3}{\Delta} = q^{-1} + 744 + 196884q + O(q^2)$$

of two modular forms of weight 12 is modular function. The $q$-coefficients $a_k$ of this function are integer.

## Lemma

*Any modular function is a polynomial in the function $j$.*

Proof. Regular at $q = 0$ modular function is modular form of the weight 0, hence is a constant.
If modular function $f(\tau) = \sum_{k=-N}^{\infty} a_k q^k$ has pole at $q = 0$ of order $N$, then the order of the pole of $(f - a_{-N} j^N)$ is less that $N$.
Induction in $N$.
Remark. If coefficients $a_k$ of modular function $f(\tau) = \sum_{k=-N}^{\infty} a_k q^k$ are integer, $f$ is polynomial in $j$ with integer coefficients.

## Proposition

$j(\tau_1) = j(_2)$ *iff* $\tau_1$ *and* $\tau_2$ *belong to the same* $\mathrm{SL}_2(\mathbb{Z})$*-orbit.*

A proof follows from the formula

$$\sum_{\tau \in \Phi} \nu_\tau(f) + \nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) = 0$$

for function $f(\tau) = j(\tau) - j(\tau_1)$.

# Some Number Theory.Kronecker's Liebste Jugendtroum.

## Proposition

*Let $\tau$ be an imaginary quadratic irrationality of the shape $\frac{m+\sqrt{D}}{2l}$, $D \equiv 0, 1 \mathrm{mod} 4$, $D < 0$, $m^2 \equiv D \mathrm{mod} 4l$. Then $j(\tau)$ is algebraic integer number.*

Proof. Such a $\tau$ is a solution of quadratic equation $c\tau^2 + (d-a)\tau - b = 0$ for $c = l$, $a = d + m$, $b = (D - m^2)/(4l)$. The last is just equation $\tau = \frac{a\tau+b}{c\tau+d}$, so $\tau$ is a fixed point of action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Pick $a = -d = m/2$ for even $D$ and $a = (m+1)/2$, $d = (-m+1)/2$ for odd $D$. Then the deteminants of these matrices are equal to $-\frac{D}{4}$ and $-\frac{D-1}{4}$ respectively.

We shall prove that $j(\tau)$ is a root of some polynomial with integer coefficient(and leading coefficient 1). We perform this by the following trick. We prove that for matrices $M$ with fixed determinant $N$ the *functions* $j(\tau)$ and $j(M(\tau))$ are algebraically dependent with integer coefficients. The restriction of this dependence to a fixed point produces the required polynomial. Definitely the set $\{M(\tau\}$ is infinite, but $j(M(\tau))$ depends in the $\mathrm{SL}_2(\mathbb{Z})$-orbit only. Denote by $\mathrm{Mat}^N$ the set of all $2 \times 2$ integer matrices with determinant $N$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts by left multiplication with finite number of orbits:

## Lemma

$$\mathrm{Mat}^N = \bigcup_{\substack{ad = N, \\ 0 \le b < d}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

So, all values $j(M(\tau))$ are the finite set $j(\frac{a\tau+b}{d})$, $ad = N, 0 \le b < d$.

The are roots of the polynomial

$$\Lambda_N(X) = \prod_{\substack{ad=N,\\ 0\le b<d}} \left(X - j(\frac{a\tau+b}{d})\right)$$

The coefficients of this polynomial are the elementary symmetric polynomials in $j(\frac{a\tau+b}{d})$.
First we prove that elementary symmetric polynomials in $j(\frac{a\tau+b}{d})$ are polynomials in $j(\tau)$ with rational coefficients.
As $M(\gamma(\tau)) = M(\gamma)(\tau)$, the induced by $\tau \to \gamma(\tau)$ action of $\mathrm{SL}_2(\mathbb{Z})$ on the set $\{j(M(\tau))\}$ reduces to right action action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{Mat}^N$. This right action commutes with the left action, so permutes left orbits. Hence, symmetric functions in $\{j(M(\tau))\}$ are modular invariant.

Pass to the $q$-expansions. Tt is easier to operate with the Newton symmetric functions $p_l(x_1, \cdots, x_n) = x_1^k + \cdots + x_n^k$, as they are just averaging of the powers of the variable; for averaging of power series $\sum_k a_k \exp(2\pi i k\tau)$ we evidently have

$$\sum_{0 \le b < d} \sum_k a_k \exp\left(2\pi i k \frac{a\tau + b}{d}\right) = \sum_j d a_{dj} \exp\left(2\pi i j a\tau\right),$$

as for $k$ not divisible by $d$

$$\sum_{0 \le b < d} \exp\left(2\pi i k \frac{b}{d}\right) = 0,$$

and for divisible $k$ this sum equals $d$.
So, the Newton symmetric functions of $j(\frac{a\tau+b}{d})$ have required $q$-expansion and are polynomials in $j(\tau)$ with integer coefficients.

The elementary symmetric functions are polynomials in the Newton functions with rational coefficients, hence are series in $\exp(2\pi i\tau)$ with rational coefficients. At the other hand, just by definition they are series in $\exp\left(2\pi i \frac{\tau}{N}\right)$ with coefficients in $\mathbb{Z}\left[\exp\left(\frac{2\pi i}{N}\right)\right]$. So they are series in $\exp(2\pi i\tau)$ with integer coefficients. Hence $\Lambda_N(X)$ equals $\Psi_N(X, j(\tau))$ where $\Psi_N(X, Y)$ is a polynomial with integer coefficients. For calculation we can use the following trick.
As $j = q^{-1} + \sum_{k=0}^{\infty} a_k q^k$, $j^{-1} = q + \sum_{k=2}^{\infty} b_k q^k$, one can invert this relation: $q = j^{-1} + \sum_{k=2}^{\infty} r_k j^{-k}$. Then
$j^{-1} = q + \sum_{k=0}^{\infty} b_k q^k . \Psi_N(X, j(\tau))$ is the result of the substitution this expression of $q$ into $\Lambda_N(X)$. Such $\Psi_N(X, Y)$ is a priori a polynomial in $X$ and a Laurent series in $Y^{-1}$.
The values of $j$, which we study, are roots of the polynomial $\Psi_N(X, X)$. We shall evaluate the coefficient of the leading term of this polynomial.
For fixed $a$ and $d$ consider the product

$$\Lambda_{a,d}(X) = \prod_{0 \le b < d} \left(X - j\left(\frac{a\tau + b}{d}\right)\right)$$

As above, $\Lambda_{a,d}(X)$ is equal to $\Psi_{a,d}(X, j(\tau))$, where $\Psi_{a,d}(X, Y)$ is a polynomial in $X$ and a Laurent series in $Y^{-1}$

$$\Lambda_{a,d}(X) = \prod_{0 \le b < d} \left( \left( X - \exp(-2\pi i \frac{a\tau + b}{d}) \right) + \phi_a \right) =$$

$$= \prod_{0 \le b < d} \left( X - \exp(-2\pi i \frac{a\tau + b}{d}) \right) + \sum_{\alpha} g_\alpha X^\alpha,$$

where $\phi_a(\tau)$ corresponds to the sum of the regular terms in expansion of $j$; $g_\alpha(\tau)$ is the sum of the products which contains at least one $\phi_a$ as a factor. So, the degree of $\phi_a$ in $\exp(-\frac{a\tau}{d})$ is less than $d - \alpha$.
$\prod_{0 \le b < d} \left( X - \exp(-\frac{a\tau + b}{d}) \right) = X^d - \left( \exp\left(-\frac{a\tau}{d}\right) \right)^d = X^d - q^{-a}$.

Restrict ourselves by the case then $N$ is not square, so either $a > d$ or $a < d$
for $a > d$ the degree of $g_\alpha$ in $\exp(-2\pi i\tau)$ is less then $(d - \alpha)\frac{a}{d}$. so the degree corespondent contribution to
For $a < d$ we apply the following speculation. The polynomial $\Psi$ is symmetric: $\Psi(X, Y) = \Psi(Y, X)$.. Indeed, If $\tau_1 = (a\tau_2 + b)/(c\tau_2 + d)$, then $\tau_2 = (d\tau 1 - b)/(-c\tau_1 + a)$. The symmetry interchanges $a$ and $d$, hence $\psi_{a,d}$ is a Laurent series in $Y$ of total degree less that $d$ for $d > a$.
We have proved that the "auxiliary" terms $g_\alpha$ does not contribute to the leading term, So the leading term of $\Psi_n(X, X) = \prod_{ad=N} \Psi_{a,d}(X, X)$ equals $\pm 1$