Elementary Introduction to the Theory of Automorphic forms
Lecture10

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The product formula for difference of the j-invariants.

Theorem

For modular invariant $j(q) = q^{-1} + \sum c(k)q^k$ one has

$$j(p) - j(q) = \frac{q-p}{qp} \prod_{m,n=1}^{\infty} (1 - p^m q^n)^{c(mn)}$$

Proof.

$$j(p) - j(q) = \frac{q-p}{qp} \left(1 - \sum_{m,n=1}^{\infty} c(m+n-1) p^m q^n \right).$$

Consider the logarithmic derivative with respect to the second variable $\frac{\log p}{2\pi i}$

$$\frac{p}{p-q} - 1 + \sum_{m,n=1}^{\infty} C(m,n) p^m q^n = \sum_{n=1}^{\infty} q^n \left(p^{-n} + \sum_{m=1}^{\infty} C(m,n) p^m \right) =$$

$$= \sum_{n=1}^{\infty} q^n \mathcal{P}_{-n}(p)$$

As $d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{d\tau}{(c\tau+d)^2}$, the logarithmic derivative should transform like modular form of the weight 2 (it is NOT modular form of the weight 2 as is singular at infinity).
Its q-coefficients $\mathcal{P}_{-n-1}(p)$ also transform like modular form of the weight 2.
From the cocycle condition we see that $(c\tau+d)^{-2}\mathcal{P}\left(\frac{a\tau+b}{c\tau+d}\right)$ depends from the left $\mathrm{SL}_2(\mathbb{Z})$ orbit of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

From a speculation like in the previous lecture we see that

$$N^{-1}T_N(\mathcal{P}_{-1})(p)=p^{-N}+\sum_{l=1}^{\infty}\left(\sum_{ad=N}d^{-1}C(dl,1)\right)p^{la}$$

transforms like modular form of the weight 2. *Remark* Usually the Hecke operators T_N are defined by analogous formulas for modular forms.

We see that the difference $\mathcal{P}_{-N}-T_N(\mathcal{P}_{-1})$ has not singularity at infinity and, so, is modular form of the weight 2. According to the structure theorem it vanishes. Substitute $T_N(\mathcal{P}_{-1})$ instead $\mathcal{P}_{-N}$. We get

$$\sum_{m,n=1}^{\infty}C(m,n)p^mq^n=\sum_{l,a,d=1}^{\infty}d^{-1}C(dl,1)p^{la}q^{ad}=$$

$$=\sum_{l,d=1}^{\infty}d^{-1}C(dl,1)d\log(1-p^lq^d).$$

Finally we get for the infinite sum

$$(j(p)-j(q))\frac{qp}{q-p} = \left(1-\sum_{m,n=1}^{\infty} c(m+n-1)p^m q^n\right) =$$

$$\prod_{l,d=1}^{\infty} \log(1-p^l q^d)^{C(ld,1)/(ld)}.$$

Compare coefficient of the monomial $p^m q$. In the product the monomial with first power of q corresponds to taking second term in exactly one factor with $d=1$, in other factors we choose the term 1. So $c(n)=C(n,1)$

So, we have proved the proposition in the following sense. This equality in the ring of the formal series in q with coefficients in convergent series in p.

Indeed, in the proof we differentiate and integrate with respect to p and use formula for geometric progression: $\frac{p}{1-pq}=\sum_{l=0}^{\infty} q^l p^{-l}$. All this operation are valid in the described above ring.

There is very natural question: is it possible to prove any version of convergence in q.

Theorem

In domain $\log|q|\log|p| > 4\pi^2$ *the product converges*

We use that if $\sum \alpha_l$ converges then $\prod(1+\alpha_l)$ converges We shall estimate coefficients of the j-invariants

Lemma

There are to possibility for $(a\tau+b)/(c\tau+d)$*: either it is equal to* $\tau+n$ *so* $\exp(2\pi i(a\tau+b)/(c\tau+d)) = \exp(2\pi i\tau)$ *or* $\Im((a\tau+b)/(c\tau+d)) \leq 1/\Im(\tau)$

Proof. The first case corresponds to $c=0$. If $c\neq 0$

$$\Im((a\tau+b)/(c\tau+d)) = \frac{\Im(\tau)}{|c\tau+d)|^2} \leq \frac{\Im(\tau)}{\Im(\tau)^2},$$

as $|c\tau+d)|^2 \geq c^2\Im(\tau)^2 \geq \Im(\tau)^2$.

Lemma

The coefficients $c(k)$ of the j-invariant $j(q) = q^{-1} + \sum c(k)q^k$ satisfy to the estimate
$|c(k)| \leq (K + \exp(\frac{-4\pi^2}{\log(r)}))r^{-n}$
for some constant K and any $\exp(-\sqrt{3}\pi) < r < 1$

Proof. These coefficients can be calculate by the Cauchy integral formula. So, it is sufficient to prove that $|j(q)| < K + \exp(\frac{4\pi^2}{\log(|q|)})$ if $\exp(-\sqrt{3}\pi) < |q| < 1$
Any τ, $\Im(\tau) \leq \sqrt{3}/2$ is an image of some τ' from the modular figure, so $\Im(\tau') \geq \sqrt{3}/2$. Denote by K the maximum of $|j(\tau) - \exp(-2\pi i\tau)|$ on the modular figure. Then if $|j(\tau)| = |j(\tau')| > M \exp(2\pi\Im(\tau') = |\exp(-2\pi i\tau')| > M - C$.
From the previous lemma

$$\Im(\tau) \leq \frac{1}{\Im(\tau')}$$

. combining these inequalities we get the bound.

Now we are able to prove the convergence theorem for the series

$$\sum_{m,n=1}^{\infty} c(mn) p^m q^n.$$

Fix some ρ such that $|q| < \rho < 1$ and $(\log|q| - \log a)\log|p| > 4\pi^2$. For any m pick $r > (|q|\rho^{-1})^{\frac{1}{m}}$.Then

$$|c(mn)p^m q^n| < (K + \exp(-\frac{4\pi^2}{\log(r)}))r^{-nm}|p|^m|q|^n =$$

$$= K(r^{-m}|q|)^n|p|^m + \exp(m\log|p| - \frac{4\pi^2}{\log(r)})(r^{-m}|q|)^n <$$

$$< K\rho^n|p|^m + \exp(m(\log|p| - \frac{4\pi^2}{\log|q| - \log(a)}))\rho^n.$$

According to inequalities on ρ this series converges.