Обобщенные ф-ции.

① $f \in L^1(\mathbb{R}^n)$

$$f_\varepsilon(x) := \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0$$

Д-ть: $$\lim_{\varepsilon \to 0^+} f_\varepsilon \overset{\mathcal{D}'(\mathbb{R}^n)}{=} \delta \cdot \int_{\mathbb{R}^n} f\,dx$$

$$\int_{\mathbb{R}^n} f_\varepsilon(x)\,dx = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} f\Big(\underbrace{\frac{x}{\varepsilon}}_{z}\Big)\,dx = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} f(z)\,\varepsilon^n\,dz = \int_{\mathbb{R}^n} f(z)\,dz.$$

$z := \frac{x}{\varepsilon}$

$x = \varepsilon z$

$dx = \varepsilon^n dz$

Dim.: $\varphi \in \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$

$$\langle \underbrace{\varphi}_{\in \mathcal{D}(\mathbb{R}^n)}, \underbrace{f_\varepsilon}_{\in \mathcal{D}'(\mathbb{R}^n)} \rangle \xrightarrow[\varepsilon \to 0^+]{} \langle \varphi, \delta \rangle \cdot \int_{\mathbb{R}^n} f\,dx$$

$$\langle \varphi, \delta \rangle \cdot \int_{\mathbb{R}^n} f\,dx = \varphi(0) \int_{\mathbb{R}^n} f\,dx.$$

$$\langle \varphi, f_\varepsilon \rangle = \int_{\mathbb{R}^n} \varphi(x) f_\varepsilon(x)\,dx$$

$$= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \varphi(x) f\left(\frac{x}{\varepsilon}\right) dx = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \varphi(\varepsilon z) f(z)\, \varepsilon^n\, dz =$$

($\frac{x}{\varepsilon} = z$)

$$= \int_{\mathbb{R}^n} \underbrace{\varphi(\varepsilon z) f(z)}\, dz \xrightarrow[\varepsilon \to 0^+]{} \int_{\mathbb{R}^n} \underbrace{\varphi(0) f(z)}\, dz.$$

$$|\varphi(\varepsilon z) f(z)| \leq C\, |f(x)| \quad \leftarrow \text{(!)} \quad f \in L^1(\mathbb{R}^n)$$

$$|\varphi(y)| \leq C \qquad \forall y \in \mathbb{R}^n$$

$n = 1$

$f(x) = \frac{1}{1+x^2}$

(2) $$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{x^2+\varepsilon^2} \underset{\mathcal{D}'(\mathbb{R})}{=} \; ?$$

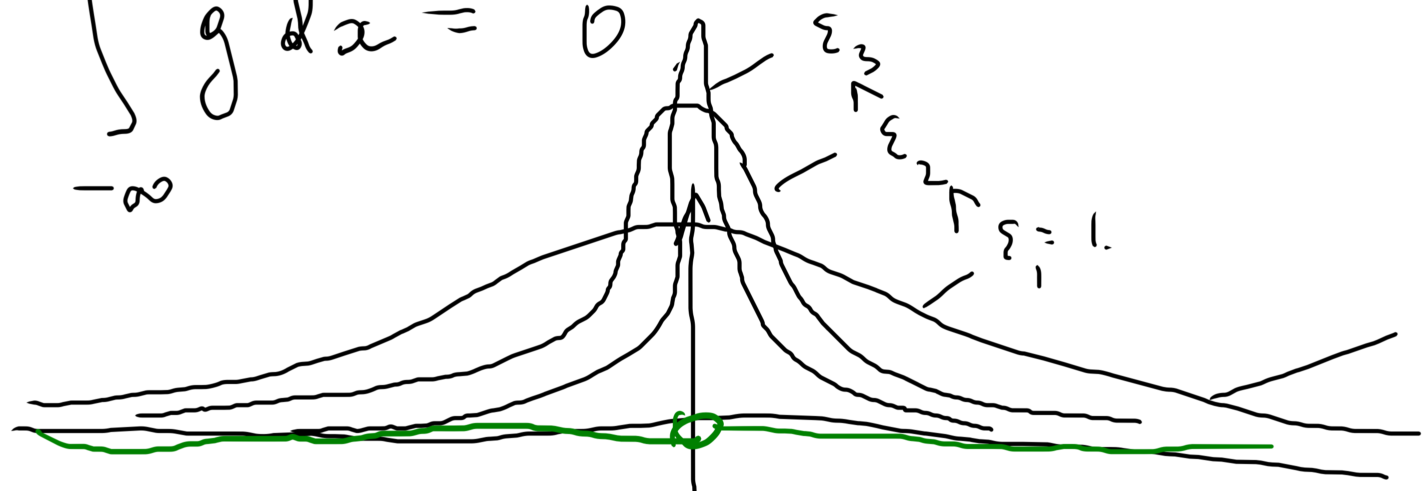
$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\underset{\in \mathbb{R}}{x^2}+\varepsilon^2} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\varepsilon^2\left(1+\frac{x^2}{\varepsilon^2}\right)} = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \frac{1}{1+\left(\frac{x}{\varepsilon}\right)^2} =$$

$$= \delta \cdot \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi\delta$$

Кстати $\frac{\varepsilon}{x^2+\varepsilon^2} \xrightarrow[\varepsilon\to 0+]{} g(x) := \begin{cases} 0, & x\neq 0 \\ +\infty & x=0\end{cases}$

$\int_{-\infty}^{+\infty} g\,dx = 0$ $\quad +\infty$

$\int_{-\infty}^{+\infty}\frac{\varepsilon\,dx}{x^2+\varepsilon^2} = \pi$



Д/З (!!) обосновать!

$f(x) = \frac{\sin x}{x}$

$\frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)$, $f(x) = \frac{\sin x}{x}$

(3) $\lim_{\varepsilon\to 0^+}\frac{1}{x}\sin\frac{x}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\,\frac{1}{x/\varepsilon}\sin x/\varepsilon = \delta\cdot\int_{-\infty}^{+\infty}\frac{\sin x}{x}\,dx = \pi\delta$

↑ $\mathcal{D}'(\mathbb{R})$

Идея: $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle \varphi, f_\varepsilon \rangle = \int_{-\infty}^{+\infty} \varphi(x) f_\varepsilon(x)\,dx = \ldots$$

(corrected in red: limits $-A$ to A)

$$= \int_{-A/\varepsilon}^{A/\varepsilon} \varphi(\varepsilon z) f(z)\,dz$$

$$f_\varepsilon(x) = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right),$$

$$f(x) = \frac{\sin x}{x}$$

$$f^k(x) = \frac{\sin x}{x} \cdot 1_{[-k, k]}(x)$$

$$\varepsilon_j \xrightarrow[j\to\infty]{} 0^+$$

$$\left\langle \varphi, \frac{1}{\varepsilon_j} f^k\left(\frac{\cdot}{\varepsilon_j}\right)\right\rangle \xrightarrow[j\to\infty]{} \varphi(0)\int_{-k}^{k}\frac{\sin x}{x}\,dx.$$

$$\downarrow k\to+\infty$$

$$\varphi(0)\cdot\int_{-\infty}^{+\infty}\frac{\sin x}{x}\,dx$$

π и.

$\exists\ \{\varepsilon_{j_k}\}$ - подпосл. $\{\varepsilon_j\}$

$$\left\langle \varphi, \frac{1}{\varepsilon_{j_k}} f^k\left(\frac{\cdot}{\varepsilon_{j_k}}\right)\right\rangle \xrightarrow[k\to+\infty]{} \pi\varphi(0)$$

$$\left\langle \varphi, \frac{1}{\varepsilon_{j_k}} f\left(\frac{\cdot}{\varepsilon_{j_k}}\right)\right\rangle$$

$$\int_{-\infty}^{+\infty} \left|\frac{\sin x}{x}\right| dx = 2\int_{0}^{+\infty} \frac{|\sin x|}{x}\, dx = +\infty.$$

Д/З (!)
вычислить этот интеграл

$f(x) = \frac{\sin x}{x}$ $\quad f \notin L^1(\mathbb{R})$

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \pi$$

несобственный интеграл Римана (!)

(4) $$\lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{2\varepsilon}} e^{-x^2/2\varepsilon} \underset{\mathcal{D}'(\mathbb{R})}{=} \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{2\varepsilon}} e^{-\left(\frac{x}{\sqrt{2\varepsilon}}\right)^2}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) = \delta \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-x^2} dx}_{=\sqrt{2}} = \sqrt{\pi}\,\delta \; ?$$

$$f(x) = e^{-x^2}$$

⑤ $\begin{cases} \varphi \in \mathcal{D}(\mathbb{R}) & \varphi(0) = 0 \\ \psi(x) = \dfrac{\varphi(x)}{x} \end{cases}$

D - ть: $\psi \in \mathcal{D}(\mathbb{R})$

D - во. $\varphi(x) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(0)}{k!} x^k + o(x^n) =$

$= \sum_{k=1}^{n} \frac{\varphi^{(k)}(0)}{k!} x^k + o(x^n)$

$\uparrow$

$\varphi(0) = 0$

$\sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} x^k + o(x^{n-1}) = \psi(x) = \frac{\varphi(x)}{x} = \sum_{k=1}^{n} \frac{\varphi^{(k)}(0)}{k!} x^{k-1} + o(x^{n-1}) = \sum_{k=0}^{n-1} \frac{\varphi^{(k+1)}(0)}{(k+1)!} x^k + o(x^{n-1})$

$$\psi^{(k)}(0) = \frac{\varphi^{(k+1)}(0)}{k+1}.$$

(6) Найти $u \in \mathcal{D}'(\mathbb{R})$,

$x u = 0$ в смысле $\mathcal{D}'(\mathbb{R})$

x — обычная функция, u — $\mathcal{D}'(\mathbb{R})$

Т.е. $\forall \varphi \in \mathcal{D}(\mathbb{R})$:

$\langle x\varphi, u\rangle =: \langle \varphi, xu\rangle = 0$

N.B. $u \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$ обл.

$a \in C^\infty(\Omega)$

$au \in \mathcal{D}'(\Omega)$

$\langle \varphi, au\rangle := \langle a\varphi, u\rangle$, $\varphi \in \mathcal{D}(\Omega)$

Решение:

$\psi \in \mathcal{D}(\mathbb{R})$

$\langle \psi, xu \rangle =$

$\langle x\psi, u \rangle = 0 \qquad (1)$

Если $u \in \mathcal{D}'(\Omega)$ удовл. (1), то

$\langle \varphi, u \rangle = 0$, если $\varphi(0) = 0 \qquad (2)$.

$\varphi \in \mathcal{D}(\mathbb{R})$.

$(1) \Leftrightarrow (2)$.

$\Updownarrow$

(3)

(т.к. $\varphi \in \mathcal{D}(\mathbb{R})$: $\varphi(0) = 0 \Rightarrow$

$\Rightarrow \varphi = x \dfrac{\varphi(x)}{x} = x\psi$.

$\dfrac{\varphi(x)}{x} = \psi \in \mathcal{D}(\mathbb{R})$)

$\varphi_1(0) = \varphi_2(0)$ $\quad\quad \varphi_{1,2} \in \mathcal{D}(\mathbb{R})$

$\langle \varphi_1, u \rangle = \langle \varphi_2, u \rangle$ $\quad (3)$

$$\boxed{\langle \varphi, u \rangle = C\varphi(0); \quad \varphi \in \mathcal{D}(\mathbb{R})}$$

(4)

$\langle \varphi, C\delta \rangle$.

Проверим (4) и найдем C

$\langle \varphi, u \rangle = \langle \varphi(0)\tilde{\varphi}, u \rangle = \varphi(0)\langle \tilde{\varphi}, u \rangle$ ($\langle \tilde{\varphi}, u \rangle = C$)

$\exists\, \varphi(0) \neq 0$; $\quad \tilde{\varphi}(0) = 1$.

$= C\varphi(0)$.

Итог: $u = C\delta$. $\qquad xu = 0$.

⑦ $u \in \mathcal{D}'(\mathbb{R})$: $\qquad x^2 u = 0$ в $\mathcal{D}'(\mathbb{R})$

Решение: $\quad x(xu) = 0 \iff xV = 0$ (где $V = xu$), $V \in \mathcal{D}'(\mathbb{R})$

$xu = V = C\delta$. $\qquad$ Итого: $xu = C\delta$

$u =$ частное (какое-то) реш. $xu = C\delta$ + общее решение $xu = 0$.

$xu = C\delta \iff \langle x\varphi, u\rangle = C\varphi(0).$

$\varphi \in \mathcal{D}(\mathbb{R})$

$u = C_1 \delta'(0)$

$\langle \varphi, u\rangle = C_1 \langle \varphi, \delta'(0)\rangle = -C_1 \varphi'(0)$

$\varphi \in \mathcal{D}(\Omega)$

$\langle x\varphi, u\rangle = -C_1 (x\varphi)'(0) = -C_1 \left(x\varphi'(x) + \varphi(x)\right)\Big|_{x=0}$

$= -C_1 \varphi(0) = -C_1 \langle \varphi, \delta\rangle =$

$= \langle \varphi, -C_1\delta\rangle$

$$-C_1 = C \Rightarrow C_1 = -C.$$

Usiamo:

$$\boxed{u = C\delta' + D\delta}$$

$$(C, D) \in \mathbb{R}^2$$