Elementary Introduction to the Theory of Automorphic forms
Lecture11

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Expansion for the Siegel Forms and the Jacobi Forms.

Reminder. The symplectic group Sp_g is the group of $2g \times 2g$ matrices h the shape $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$ is inverse of h. So, $\mathrm{SL}_2 = \mathrm{Sp}_1$.
The Siegel upper half-space $\mathbb{H}_g$ is the set of complex symmetric $g \times g$ matrices Ω with positive defined imaginary part, $\Omega^t = \Omega, \frac{\Omega - \overline{\Omega}}{2i} \gg 0$.
The group $\mathrm{Sp}_g(\mathbb{R})$ acts on $\mathbb{H}_g$ by the rule $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$.
The automorphic factor is equal to $\det(C\Omega + D)$. The Siegel modular form $F(\Omega)$ of the weight k is a holomorphic function in Ω such that

$$F\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^k F(\Omega)$$

For genus 2 Sigel upper half-plane put

$$\Omega = \begin{pmatrix} \tau & \xi \\ \xi & \omega \end{pmatrix}$$

Note that if $C = 0$ $D^{-1} = A^t$ and the action is $\Omega \mapsto (A\Omega + B)A^t$

A	B	$(A\Omega + B)A^t$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \tau & \xi \\ \xi & \omega + 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \tau & \xi + 1 \\ \xi + 1 & \omega \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \tau & \xi + \tau \\ \xi + \tau & \omega + 2\xi + \tau \end{pmatrix}$

The automorphic factor is equal to $\det A^{-1} = 1$.

According to the first row of the table we can pass to variable $p = \exp(2\pi i\omega)$. Hence any Siegel modular form $F(\Omega)$ of the weight k is a function on the domain $(\tau \in \mathbb{H}, \xi \in \mathbb{C}, \log 0 < |p| < -\max\left(0, \frac{(\Im(\xi))^2}{\Im(\tau)}\right)$, so can be expanded as $F(\Omega) = \sum_m \phi_m(\tau, \xi) p^m$, as . From the second and third rows we deduce that $\phi_m(\tau, \xi + 1) = \phi_m(\tau, \xi)$, $\phi_m(\tau, \xi + \tau) = \exp(-2\pi i m(2\xi + \tau))\phi_m(\tau, \xi)$

If $ad - bc = 1$, put $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$, $= \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ Then $(\tau, \xi, \omega) \to \left(\frac{a\tau+b}{c\tau+d}, \frac{\xi}{c\tau+d}, \omega - \frac{c\xi^2}{c\tau+d}\right)$. The automorphic factor is equal to $c\tau + d$. So

$$\phi_m\left(\frac{a\tau + b}{c\tau + d}, \frac{\xi}{c\tau + d}\right) = (c\tau + d)^k exp\left(2\pi i m\frac{c\xi^2}{c\tau + d}\right)\phi_m(\tau, \xi).$$

Such functions ϕ_m are known as weak *Jacobi modular forms* of the weight k and *index m*.

By the argument principle applied to the parallelogram in ξ complex plane with vertices 0, 1,$1+\tau$ and τ we deduce that the number of zeroes minus number poles equals to $2m$, So for holomorphic F only non negative m contribute to p-expansion. At the other hand, from the first and second rows of the table and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one has $F(\Omega) = \sum c(n,r,m) q^n z^r p^m$, where, as above $q = \exp(2\pi i \tau)$, $z = \exp(2\pi i \xi)$ and $p = \exp(2\pi i \omega)$.This expansion is not so straightforward as $(\exp(2\pi i \tau), \exp(2\pi i \xi), \exp(2\pi i \omega))$ maps $\mathbb{H}_2$ to the domain $0 < |q| < 1, 0 < |p| < 1, 0 < |z| < \sqrt{|q||p|}$. It is useful to express $q^n z^r p^m = \exp(2\pi i \mathrm{tr}(M\Omega))$, where $M = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$.

from the previous slide we now that in this sum terms with negative m and, by symmetry, with negative n are absent. Let as prove using action $\Omega \mapsto A\Omega A^t$ the following

Proposition

A monomial contributes to the expansion of a modular form only if the matrix M is positive semidefined, so $4mn - r^2 \geq 0$.

Proof. The monomial $\exp(2\pi i \mathrm{tr}(M\Omega))$ transforms to $\exp(2\pi i \mathrm{tr}(M(A\Omega A^t))) = \exp(2\pi i \mathrm{tr}((A^t MA)\Omega))$, so the coefficients corresponding to M and $A^t MA$ coincide.
Let M is not positive semidefined, then there is an integer primitive $(\gcd(\alpha, \gamma) = 1)$ column v $v^t = \begin{pmatrix} \alpha & \gamma \end{pmatrix}$ such that $v^t Mv < 0$.

Complete it to unimodular matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.Then one of the diagonal element of the matrix $A^t MA$ equals $v^t Mv$ and is negative, so corresponding to $A^t MA$ coefficient vanishes and coefficients of M-monomial vanishes too. In contrast with dimension one, the positivity condition follows from the automorphic property. This motivate stronger version of the definition of Jacobi form

Definition

(The Theory of Jacobi Forms Birkhäuser Basel Martin Eichler, Don Zagier Introduction) A function $F(\tau, \xi)$ *on* $\mathbb{H} \times \mathbb{C}$ *is Jacobi modular form of the weight k and index m if it is weak Jacobi form:*
$\phi_m(\tau, \xi + 1) = \phi_m(\tau, \xi)$,
$\phi_m(\tau, \xi + \tau) = \exp(-2\pi i m(2\xi + \tau))\phi_m(\tau, \xi)$ *If* $ad - bc = 1$,
$\phi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{\xi}{c\tau+d}\right) = (c\tau + d)^k exp\left(2\pi i m\frac{c\xi^2}{c\tau+d}\right)\phi_m(\tau, \xi)$.
and can be represented as a sum

$$F(\tau, \xi) = \sum_{n \geq 0, 4mn - r^2 \geq 0} a(n, r)\exp(2\pi i(n\tau + r\xi)).$$

The Jacobi Group

$$\begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \varkappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Interchange 1 and 2

$$h_{\mathrm{ell}} = \begin{pmatrix} 1 & \lambda & \mu & \varkappa \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, h_{\mathrm{mod}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Evidently the product of two matrices of elliptic type is elliptic, the product of two matrices of modular type is modular. The product

$$h_{\rm mod}h_{\rm ell} = \begin{pmatrix} 1 & \lambda & \mu & \varkappa \\ 0 & a & b & a\mu - b\lambda \\ 0 & c & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & a\lambda' + c\mu' & d\lambda' + d\mu' & \varkappa \\ 0 & a & b & \mu' \\ 0 & c & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix} = h'_{\rm ell}h_{\rm mod}$$

$$\begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} \Longleftrightarrow (\lambda \quad \mu) = (\lambda' \quad \mu') \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Matrices of such shape forms a group. Indeed
$(h^{(1)}_{\rm mod}h^{(1)}_{\rm ell})(h^{(2)}_{\rm mod}h^{(2)}_{\rm ell}) = h^{(1)}_{\rm mod}(h^{(1)}_{\rm ell}h^{(2)}_{\rm mod})h^{(2)}_{\rm ell} =$
$h^{(1)}_{\rm mod}(h^{(2)}_{\rm mod}{h^{(1)}}'_{\rm ell})h^{(2)}_{\rm ell} = (h^{(1)}_{\rm mod}h^{(2)}_{\rm mod})({h^{(1)}}'_{\rm ell}h^{(2)}_{\rm ell}) = h_{\rm mod}h_{\rm ell}$