Elementary Introduction to the Theory of Automorphic forms
Lecture12

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The Integer Jacobi Group.

$$\begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \varkappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Interchange 1 and 2

$$h_{\mathrm{ell}} = \begin{pmatrix} 1 & \lambda & \mu & \varkappa \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, h_{\mathrm{mod}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Evidently the product of two matrices of elliptic type is elliptic, the product of two matrices of modular type is modular. The product

$$h_{\mathrm{mod}} h_{\mathrm{ell}} = \begin{pmatrix} 1 & \lambda & \mu & \varkappa \\ 0 & a & b & a\mu - b\lambda \\ 0 & c & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & a\lambda' + c\mu' & d\lambda' + d\mu' & \varkappa \\ 0 & a & b & \mu' \\ 0 & c & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix} = h'_{\mathrm{ell}} h_{\mathrm{mod}}$$

$$\begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} \Longleftrightarrow (\lambda \quad \mu) = (\lambda' \quad \mu') \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Matrices of such shape forms a group. Indeed
$(h^{(1)}_{\mathrm{mod}} h^{(1)}_{\mathrm{ell}})(h^{(2)}_{\mathrm{mod}} h^{(2)}_{\mathrm{ell}}) = h^{(1)}_{\mathrm{mod}}(h^{(1)}_{\mathrm{ell}} h^{(2)}_{\mathrm{mod}})h^{(2)}_{\mathrm{ell}} =$
$h^{(1)}_{\mathrm{mod}}(h^{(2)}_{\mathrm{mod}} h^{(1)\prime}{}_{\mathrm{ell}})h^{(2)}_{\mathrm{ell}} = (h^{(1)}_{\mathrm{mod}} h^{(2)}_{\mathrm{mod}})(h^{(1)\prime}{}_{\mathrm{ell}} h^{(2)}_{\mathrm{ell}}) = h_{\mathrm{mod}} h_{\mathrm{ell}}$

The Algebraic Jacobi Group.

Evidently we can treat the entries of matrices above as elements of any commutative ring with unit. So we can determine corresponding algebraic group $\mathbb{G}^J$. It is the contain the block-diagonal subgroup SL_2 of modular transformations and block-upper triangular normal subgroup $\mathbb{G}^{\mathrm{Heis}}$,$G^{\mathrm{J}}/G^{\mathrm{Heis}} = \mathrm{SL}_2$. The Heisenberg group G^{Heis} is the central extension of the quotient group is $\mathbb{V} = \mathbb{G}_a \times \mathbb{G}_a$ with coordinates λ, μ by the additive subgroup $C = \mathbb{G}_a$ with coordinate determined by 2-cocycle $\lambda_1\mu_2 - \lambda_2\mu_1$:
$(\lambda_1, \mu_1; \varkappa_1) * (\lambda_2, \mu_2; \varkappa_2) = (\lambda_1 + \lambda_2, \mu_1 + \mu_2; \varkappa_1 + \kappa_2 + \lambda_1\mu_2 - \lambda_2\mu_1)$.
SL_2 acts on $\mathbb{V}$ as multiplication of column $\begin{pmatrix} \mu \\ -\lambda \end{pmatrix}$ by matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.This action preserves the cocycle, so SL_2 acts on $\mathbb{G}^{\mathrm{Heis}}$.

More Invariant Description

As we switch two fist basic vectors, we deal with symplectic four-dimensional space T with antidiagonal form:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The Jacobi group $\mathbb{G}^J$ stabilises the vector $v = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^t$, which generate one-dimensional subspace V. Prove that this subgroup coincides with stabiliser of this vector.

Let h stabilize vector v, this forced to vanish all except the first entries of the first column. Symplectic operator preserve the orthogonal space $V^{\perp} = \{\begin{pmatrix} * & * & * & 0 \end{pmatrix}^t\}$, so all except the last elements of the last raw vanish also.

As $T/V^{\perp}$ is dual to V, action on it is identical. This is 1 as the last diagonal element As this group preserves the flag $V \subset V^{\perp}$, hence it acts symplectic on the quotient $U = V^{\perp}/V$. This is the middle diagonal block of the matrix.

If h is identical on $V^{\perp}/V$, the operator $H = h - \mathrm{Id}$ vanishes on $T/V^{\perp}$ and $V^{\perp}/V$, so it induces maps $H_1 : V^* \to U$ and $H_2 : U \to V$. The fist map H_1 determines a vector $u \in U$, the second map H_2 get a covector $\chi \in U^*$, this covector is opposite to the sympletcic pairing in U with u.

The action of the Real Jacobi Group

For fixed τ the real Heisenberg group $G^{\mathrm{Heis}}(\mathbb{R})$ transitive acts on $\mathbb{C}$ through its quotient $\mathbb{V}(\mathbb{R})$: $\xi \to \xi + \lambda\tau + \mu$.
This action is compatible with the action of $\mathrm{SL}_2(\mathbb{R})$: $\xi \to \frac{\xi}{c\tau+d}$.
This defines transitive action of $G^{\mathrm{J}}(\mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$. The stabilizer of the point $(i, 0)$ equals to $\mathrm{SO}_2(\mathbb{R}) \times C(\mathbb{R})$. This group is not compact, so it is reasonable to pass to quotients $C(\mathbb{R})/C(\mathbb{Z})$ and $\tilde{G}^{\mathrm{J}}(\mathbb{R}) = G^{\mathrm{J}}(\mathbb{R})/C(\mathbb{Z})$
There are two automorphic factors. The first is induced by the map $G^{\mathrm{J}}/G^{\mathrm{Heis}} \to \mathrm{SL}_2$. The second is defined on G^{Heis} by the formula $\exp(2\pi i(2\lambda\xi\lambda^2\tau + \lambda\mu + \varkappa))$ and equls to $\exp\left(-2\pi i\left(2\frac{c\xi^2\tau}{c\tau+d}\right)\right)$ on SL_2. Note that the action does not depend in the center $C(\mathbb{C})$ of the group, but the second automorphic factor depends in $C(\mathbb{C})$ and can be pushed forward to the quotient by $C(\mathbb{Z})$. Hence one can define the Eisenstein series a as basic examples of the Jacobi modular forms

Several General Remarks.

In our study $p.q, r$-development we use the group of matrices of the shape $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}$ They form the stabiliser of the 2-plane V generated by to vectors $v_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^t$ and $v_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^t$. This is Lagrangian plane – the restriction of the symplectic structure on it vanishes.
In both cases the groups are *parabolic subgroups*. For the Siegel spaces of bigger dimension one can pick stabilisers of *isotropic* subspaces of appropriate dimension.
For hyper-lorenzian quadratic forms one have the analogous construction for stabiliser of an integer light-vector.

The Golden Ratio Order. Reminder.

The golden ratio is defined by expression $\phi = \frac{\sqrt{5}+1}{2}$. Consider the group $\mathcal{O}_5 = \{m\phi + n | m, n \in \mathbb{Z}\}$ This group is a ring as $\phi^2 = \phi + 1$. There are two ring embedding $\mathcal{O}_5 \ni \nu \to \nu^{\pm} \in \mathbb{R}$ of $\mathcal{O}_5$ into $\mathbb{R}$ by choosing the sign of $\sqrt{5}$: $\mathcal{O}_5 \ni \phi \to \phi^{\pm} = \frac{\pm\sqrt{5}+1}{2} \in \mathbb{R}(\sqrt{5} > 0)$ The image of any from both these maps is dense, but the image of their sum $\mathcal{O}_5 \ni \nu \to (\nu^+, \nu^-) \in \mathbb{R} \oplus \mathbb{R}$ is dense as images of 1 and ϕ form a basis: $\begin{vmatrix} 1 & \phi^+ \\ 1 & \phi^- \end{vmatrix} = -\sqrt{5} \neq 0$. Define the trace and the norm on $\mathbb{R} \oplus \mathbb{R}$ as $\mathrm{tr}(x^+, y^-) = x^+ + x^-, \mathrm{Norm}(x^+, x^-) = x^+ x^-$ and induce them to $\mathcal{O}_5$ using the canonical embedding. Call $(x^+, x^-) \in \mathbb{R} \oplus \mathbb{R}$ *totally positive* ($(x^+, x^-) \gg 0$) iff both components are positive. This notion also can be induced to $\mathcal{O}_5$. F,e. $\phi^2 = \frac{\sqrt{5}+3}{2}$ is totally positive but $\phi = \frac{\sqrt{5}+1}{2}$ is not.

We shall describe $\mathcal{O}_5$-invariant exponential functions on $\mathbb{R}^2$. For this note that the pairing $(\frac{1}{\sqrt{5}}\mathcal{O}_5) \times \mathcal{O}_5 \to \mathbb{Q}, (\mu, \nu) \to \mathrm{tr}(\mu\nu)$ is integer and perfect. So, all invariant exponential functions have the shape $e_\mu(x) = \exp(2\pi i \mathrm{tr}(\mu x))$
In contrast to the ring of the integers $\mathbb{Z}$, the group $\mathcal{O}_5^*$ of invertible elements of $\mathcal{O}_5$ does not reduce to ± 1, f.e. as $\phi(\phi - 1) = 1$, ϕ is invertible. It not so hard to check that $\mathcal{O}_5^* = \pm\phi^{\mathbb{Z}}$