Elementary Introduction to the Theory of Automorphic forms
Lecture13

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Some Geometry. Compactification.

From the geometric viewpoint it is natural to pass to the set of $\mathrm{SL}_2(\mathbb{Z})$-orbits in $\mathbb{H}$. The first approximation of this set is the modular figure $\Phi = \{\tau | |\tau|^2 \geq 1, -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}\}$. as another imitation of this set of orbit one can use another fundamental domain $\tilde{\Phi} = \{\tau | 0 < \Re(\tau) < \frac{1}{2}, |\tau - 1| > 1\}$. Indeed, cut $\Phi = \Phi^+ \cup \Phi$, $\Phi^\pm = \Phi \cap \{\pm\Re(\tau) > 0\}$. Then $\tilde{\Phi} = \Phi^+ \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^-$.

Evidently, Φ is not compact. For compactification of the set of orbits we use the following speculation. As we have mentioned in the Lecture 10, there are to possibility for $(a\tau + b)/(c\tau + d)$: either it is equal to $\tau + n$ or $\Im\left((a\tau + b)/(c\tau + d)\right) \leq 1/\Im\left(\tau\right)$. Hence for the domain $\mathcal{N} = \{\tau | \Im(\tau) > 1\}$ and $h \in \mathrm{SL}_2(\mathbb{Z})$ there are two possibilities: either $h\mathcal{N} \cup \mathcal{N} = \emptyset$, or $h\mathcal{N} = \mathcal{N}$ and $h(\tau) \in \tau + \mathbb{Z}$. SO, the set of orbits intersecting $\mathcal{N}$ coicides with set of $\mathbb{Z}$-orbits in $\mathcal{N}$. The exponential map $\exp 2\pi i$ identifies the last set of orbits with punctured disc $0 < q < \exp(-2\pi)$. One can compactify by adding the point 0

Subsets and the Level Structure

We discuss two examples of curves in surfaces. The first example is motivated by the Lecture 9. For prime p consider the subset $\tilde{T}_p$ of $\mathbb{H} \times \mathbb{H}$ define by condition $\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}, ad - bc = p$; this set is preserved by action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. We shall describe the set of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$-orbits in $\tilde{T}_p$.

Lemma(the Smith normal form) Any matrix with determinant p can be represented as $h_l^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} h_r$, where $h_l, h_r \in \mathrm{SL}_2(\mathbb{Z})$

Proof. Consider the $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$-orbit $\{h_l M h_r^{-1} | h_l, h_r \in \mathrm{SL}_2(\mathbb{Z})\}$ of some matrix M. The set of all nonzero coefficients of this matrices contains an element with minimal absolute value. This is element of some matrix,consider the set of all matrices containing this element. we can assume that this element is the first diagonal element. As left and right multiplication by triangular matrices realise the division with the reminder, from minimality me can conclude that the matrix can be chosen diagonal, as the determinant is p, the diagonal elements are p and 1.

Let two points τ and $\tilde{\tau}$ determine the same orbit , so $\tilde{\tau} = \frac{a_1\tau+b_1}{c_1\tau+d_1}, p\tilde{\tau} = \frac{a_2p\tau+b_2}{c_2p\tau+d_2}$. For generic τ we conclude that $b_1 = pb_2$, so this forces to consider the congruence subgroup $\Gamma^0(p)$ of matrices with b divisible by p

The second example is related with the Jacobi group (Lectures 11-12). Consider the embedding $\mathbb{H} \to \mathbb{H} \times \mathbb{C}$: $\tau \to (\tau, \tau/n)$ for some natural n. We shall describe the image of this map after passing to the orbits of the Jacobi group. Two points τ and $\tilde{\tau}$ belong to the one orbit iff

$$\tilde{\tau} = \frac{a\tau+b}{c\tau+d}, \frac{\tilde{\tau}}{n} = \frac{1}{c\tau+d}\left(\frac{\tau}{n} + \lambda\tau + \mu\right)$$

So, $a = 1 + n\lambda$, $b = n\mu$ and we get the group $\Gamma^1(n)$.

The Level Structure. Compactification and Modular Forms

The compactification of the set of orbits of the congruence subgroup is slightly different. First, the exponential function depends in intersection of the congruence subgroup with stabiliser of infinity. Second, note that $\mathrm{SL}_2(\mathbb{Z})$ acts transitive on $\mathbb{Q} \cup \infty$ so any rational point get the locus of non-compactness. $\mathrm{SL}_2(\mathbb{Z})$ acts on them transitive, so it is sufficient to describe the compactification near ∞. Genericlly the congrunce subgroup does not acts transitive, so the fundamental domain has several components near real line so we should compactify each such component separately. Such compactification get conditions to modular form.