Elliptic Integrals and Elliptic Functions

Introduction

§0.1 What are elliptic functions?

Elliptic function = Doubly periodic meromorphic function on $\mathbb{C}$.

Too simple object?

Indeed, in most of modern textbooks on complex analysis, elliptic functions appear usually just as examples.

BUT! in fact, many branches of modern mathematics, especially algebraic geometry, came out of study of elliptic functions in the XIXth century.

§0.2 Why are they called "elliptic functions"?

Inverse functions of elliptic integrals are elliptic functions.

Elliptic integrals:

$$z(u) = \int_{x_0}^{u} R(x, \sqrt{\varphi(x)})\, dx.$$

$R(x, s)$: rational function in two variables.

$\varphi(x)$: polynomial of degree three or four.

Example: arc length of an ellipse.

The function $u(z)$ inverse to the elliptic integral $z(u)$

is *an elliptic function*! (discovery of Abel, Jacobi, Gauss)

§0.3 Where do elliptic integrals live?

Because of the square root, integrands of elliptic integrals are "multi-valued". $\Longrightarrow$ should be considered on a Riemann surface.

- Elliptic curve

Compactification of the Riemann surface of $\sqrt{\varphi(x)}$ = elliptic curve (torus)

Example: $u(z) :=$ inverse of $\displaystyle\int^z \frac{dx}{\sqrt{\varphi(x)}}$.

Periods of an elliptic function $= \displaystyle\int_{\text{loop on the elliptic curve}} \frac{dx}{\sqrt{\varphi(x)}}$.

Elliptic curve $\cong \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. ($\omega_1, \omega_2$: periods.): Abel-Jacobi theorem

§0.4 What properties do elliptic functions have?

- Similarity of elliptic and rational functions.

Rational function = meromorphic function on the Riemann sphere.

$\Longleftrightarrow$

Elliptic function = meromorphic function on an elliptic curve.

$$\text{Rational function} = \frac{\text{polynomial}}{\text{polynomial}} \Longleftrightarrow \text{Elliptic function} = \frac{\theta\text{-function}}{\theta\text{-function}}.$$

- Similarity of elliptic and trigonometric functions.

Addition formulae:

Trigonometric: $\sin(x+y) = \sin x \cos y + \sin y \cos x$.

Elliptic: $\operatorname{sn}(x+y) = \dfrac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$. ($k$: parameter)

f has an algebraic addition formula.

$\Longleftrightarrow$ f is a rational or a trigonometric or an elliptic funtion.

Differential equations:

Trigonometric: $(\sin x)' = \cos x$, $((\sin x)')^2 = 1 - \sin^2 x$.

Elliptic: $(\operatorname{sn} x)' = \operatorname{cn} x \operatorname{dn} x$, $((\operatorname{sn} x)')^2 = (1 - \operatorname{sn}^2 x)(1 - k^2 \operatorname{sn}^2 x)$.

$((\wp(z))')^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$. ($g_2$, g_3: parameters)

§0.5 Topics related to elliptic functions.

• Algebraic geometry and number theory

The image of $\mathbb{C} \ni z \mapsto (x = \mathrm{sn}(z), y = \mathrm{sn}'(z)) \in \mathbb{C}^2$

$=$ an *algebraic curve* $y^2 = \varphi_{\mathrm{sn}}(x) := (1 - x^2)(1 - k^2 x^2)$.

The image of $\mathbb{C} \ni z \mapsto (x = \wp(z), y = \wp'(z)) \in \mathbb{C}^2$

$=$ an *algebraic curve* $y^2 = \varphi_{\wp}(x) := 4x^3 - g_2 x - g_3$.

Elliptic curve in algebraic geometry: completion of $\{(x, y) \mid y^2 = \varphi(x)\}$.

($\varphi(x) =$ polynomial in x of degee three or four).

In general (especially in number theory): $\varphi(x) \in K[x]$, $(x, y) \in K^2$.

K: a field, not necessarily $= \mathbb{C}$.

• <u>When are two elliptic curves the “same”?</u>

Elliptic curves $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\mathbb{C}/\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are the same (isomorphic)

$\Longleftrightarrow \exists a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ \dfrac{\omega_2'}{\omega_1'} = \dfrac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1}.$

$\Longrightarrow$ The theory of the *moduli space* of elliptic curves.

Elliptic curves $y^2 = 4x^3 - g_2x - g_3$ and $y^2 = 4x^3 - g_2'x - g_3'$ (over $\mathbb{C}$)

are the same (isomorphic)

$$\Longleftrightarrow j(g_2, g_3) = j(g_2', g_3'), \text{ where } j(g_2, g_3) = \frac{(12g_2)^3}{g_2^3 - 27g_3^2}.$$

$\Longrightarrow$ The theory of the *modular functions*.

§0.6 Are elliptic integrals and elliptic functions useful?

- Applications to physics.
  - Mechanics:
    - motion of pendulum.
    - shape of skipping ropes.
    - top (= rotating rigid body with one fixed point).
  - Integrable systems:
    - solutions of Korteweg-de Vries equation, Toda lattice.
    - definition of eight vertex model.

- Applications to mathematics.

  - arithmetic-geometric mean: $a \geq b > 0$, $a_0 = a$, $b_0 = b$,

$$a_{n+1} := \frac{a_n + b_n}{2}, b_{n+1} := \sqrt{a_n b_n} \Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =: M(a,b)$$

    $M(a,b)$ is expressed by means of an elliptic integral (Gauss).

    $\Longrightarrow$ A fast converging recurrent formula for π can be derived.

  - A formula for roots of quintic equations:

    Abel and Galois proved that "quintic equations are not solvable", i.e.,

    $\nexists$ formula for roots, IF one can use only $\pm, \times, /$ and $\sqrt[n]{\ }$.

    $\longleftrightarrow$ IF one can use elliptic integrals (and θ-functions),

    $\exists$ a formula for roots of quintic equations!

§0.7 Plan of the course

1. Elliptic integrals (over $\mathbb{R}$).
   (a) Arc length of ellipses.
   (b) Applications to physics (pendulum).
   (c) Classification of elliptic integrals.
2. Elliptic functions (over $\mathbb{R}$)
   (a) Inverse function of elliptic integrals.
   (b) Jacobi's elliptic functions.
   (c) Properties (addition formulae, differential equations).

3. Complex elliptic integrals.
   (a) Riemann surface of irrational (algebraic) functions.
   (b) Differentials (1-forms) and elliptic integrals.
   (c) Compactifications of Riemann surfaces of $y^2 = \varphi(x)$, $\deg \varphi = 3, 4$, i.e., elliptic curves.
   (d) Periods and complete elliptic integrals.
   (e) Abel-Jacobi theorem.
4. Elliptic functions (over $\mathbb{C}$)
   (a) Doubly periodic meromorphic functions.
   (b) Weierstrass's $\wp$-function and its properties.
   (c) Theta functions.
   (d) Complex Jacobi functions and their properties.