Elliptic Functions

Elliptic integrals over $\mathbb{R}$

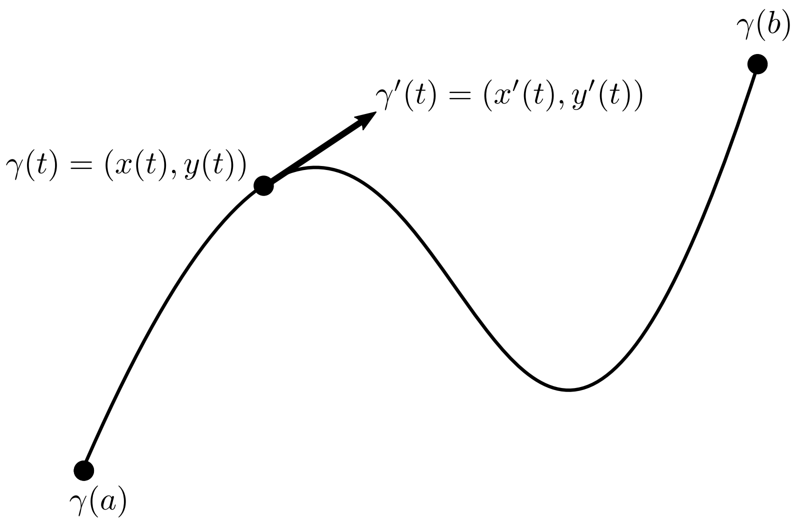
§1.1 Arc length of an ellipse

Everybody has learned that the length of a circle of radius a is $2\pi a$.

How can one *prove* this?

From the analysis course we know the formula for the arc length of a curve:

$\gamma : [a, b] \ni t \mapsto \gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$: a smooth curve.

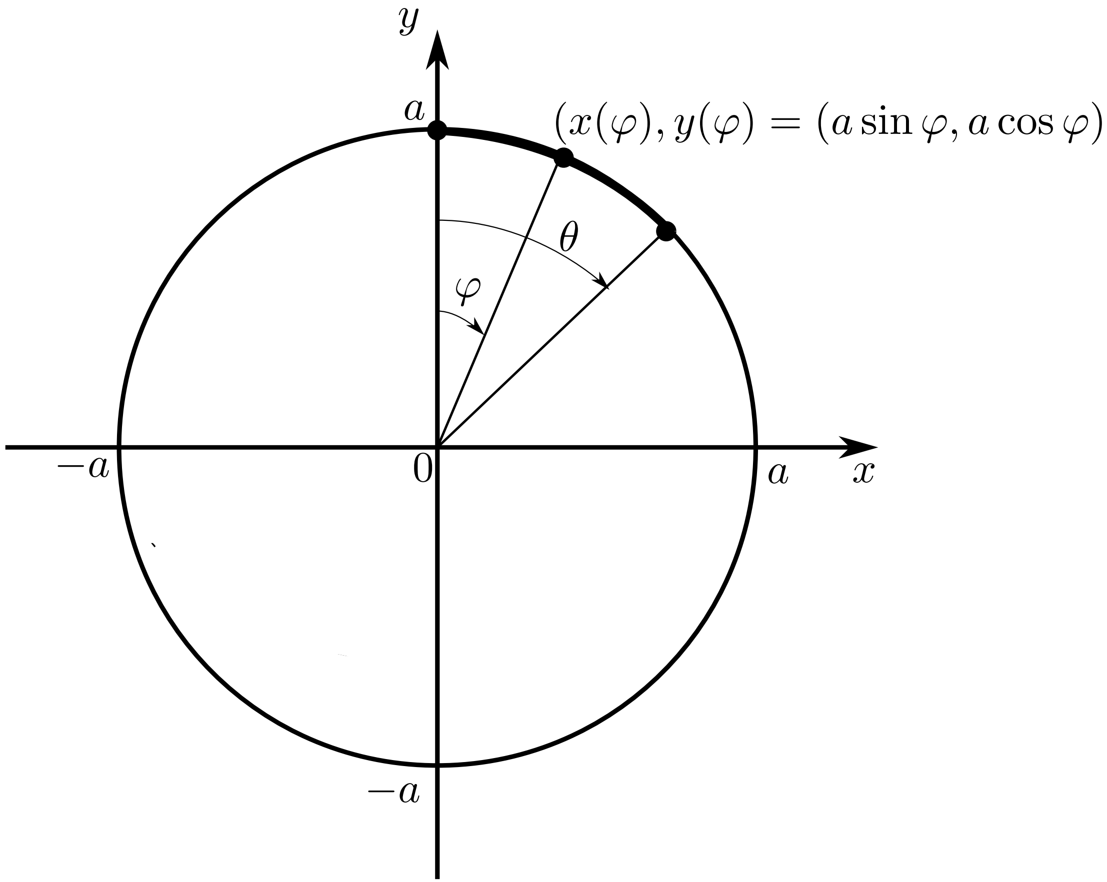


($x(t), y(t)$: C^1-class, i.e., $x'(t), y'(t)$ exist and are continuous.)

$\Longrightarrow$ the length of $\gamma = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

(The integrand $=$ speed of the moving point $(x(t), y(t))$.)

An arc of a circle: $(x(\varphi), y(\varphi)) = (a\sin\varphi, a\cos\varphi)$, $(\varphi \in [0, \theta])$.

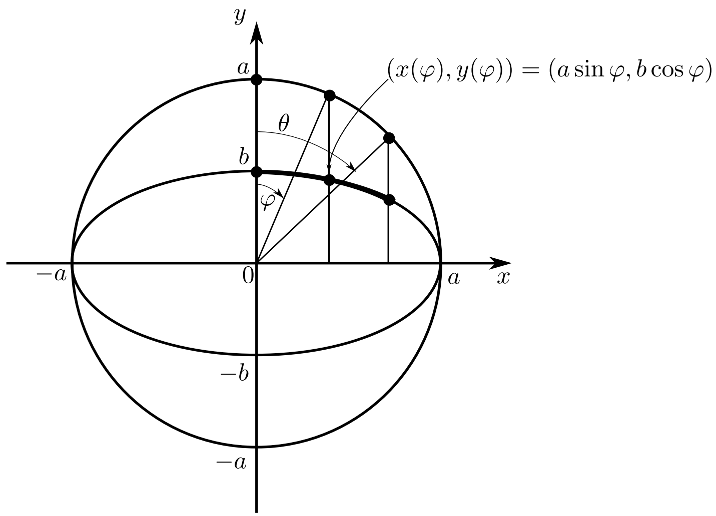


$$\text{The length of this arc} = \int_0^\theta \sqrt{\left(\frac{d}{d\varphi} a \sin\varphi\right)^2 + \left(\frac{d}{d\varphi} a \cos\varphi\right)^2}\, d\varphi$$
$$= \int_0^\theta \sqrt{a^2 \cos^2\varphi + a^2 \sin^2\varphi}\, d\varphi = a\theta.$$

In particular the arc length of the circle $= a \times 2\pi$. □

How about the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$?

Parametrisation: $(x(\varphi), y(\varphi)) = (a\sin\varphi, b\cos\varphi)$, $(\varphi \in [0, \theta])$.



$$\text{The length of this arc} = \int_0^\theta \sqrt{\left(\frac{d}{d\varphi}a\sin\varphi\right)^2 + \left(\frac{d}{d\varphi}b\cos\varphi\right)^2}\,d\varphi$$

$$= \int_0^\theta \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\,d\varphi$$

$$= a\int_0^\theta \sqrt{1 - \frac{a^2-b^2}{a^2}\sin^2\varphi}\,d\varphi$$

$$= a\int_0^\theta \sqrt{1 - k^2\sin^2\varphi}\,d\varphi.$$

$k := \sqrt{\dfrac{a^2-b^2}{a^2}}$: *modulus* of the elliptic integral, *eccentricity* of the ellipse.

$$E(k,\theta) := \int_0^\theta \sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

— incomplete *elliptic integral* of the second kind.

The length of the arc $(0 \le \varphi \le \theta) = aE\left(\sqrt{\frac{a^2-b^2}{a^2}}, \theta\right)$.

$$E(k) := E\left(k, \frac{\pi}{2}\right) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

— complete *elliptic integral* of the second kind.

The length of the ellipse $= 4aE\left(\sqrt{\frac{a^2-b^2}{a^2}}\right)$.

Except for the case $a = b$ (i.e., circles), such an integral cannot be expressed in terms of elementary functions.

(That's why we didn't learn this formula in schools!)

- Another expression of the elliptic integral of the second kind

Let us compute the arc length using the parametrisation:

$$(x, y(x)) = \left(x, b\sqrt{1 - \frac{x^2}{a^2}}\right). \qquad (x \in [0, a\cos\theta])$$

$$\begin{aligned}
\text{The arc length} &= aE(k, \theta) \\
&= \int_0^{a\sin\theta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{a\sin\theta} \sqrt{1 + \frac{b^2}{a^2}\frac{(x/a)^2}{1-(x/a)^2}}\, dx \\
&= a\int_0^{\sin\theta} \sqrt{\frac{1-k^2z^2}{1-z^2}}\, dz. \qquad (z = x/a)
\end{aligned}$$

In particular,

$$E(k,\theta) = \int_0^{\sin\theta} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz,$$

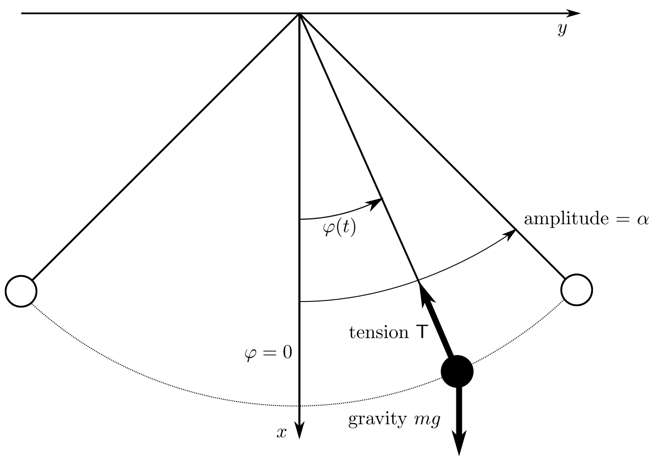
$$E(k) = \int_0^{1} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz.$$

Exercise:

(i) Find the arc length of other quadratic curves, i.e., of a parabola and a hyperbola. Which of them is expressed by an elliptic integral?

(ii) Express the arc length of the graph of $y = b\sin\frac{x}{a}$ in terms of the elliptic integral of the second kind. What arc correspond to $E(k)$?

§1.2 Motion of a simple pendulum



l: length of the string, T: tension, mg: gratity.

$(x(t), y(t)) = (l\cos\varphi(t), l\sin\varphi(t))$: coordinates of the point mass.

$$\begin{aligned}
\text{acceleration} &= \frac{d^2}{dt^2}(l\cos\varphi(t), l\sin\varphi(t)) \\
&= \frac{d}{dt}(-l\dot{\varphi}\,\sin\varphi, l\dot{\varphi}\,\cos\varphi) \\
&= (-l\ddot{\varphi}\,\sin\varphi - l\dot{\varphi}^2\,\cos\varphi, l\ddot{\varphi}\,\cos\varphi - l\dot{\varphi}^2\,\sin\varphi)
\end{aligned}$$

The equation of motion:

$$ml \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \ddot{\varphi} - ml \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} \dot{\varphi}^2 = \begin{pmatrix} -T\cos\varphi + mg \\ -T\sin\varphi \end{pmatrix}.$$

x-component $\times(-\sin\varphi)+$ y-component $\times\cos\varphi$: $ml\ddot{\varphi} = -mg\sin\varphi$, i.e.,

$$\frac{d^2\varphi}{dt^2} = -\omega^2\sin\varphi, \qquad \omega := \sqrt{\frac{g}{l}}.$$

Linear approximation: amplitude $\ll 1 \Longrightarrow \sin\varphi \approx \varphi$.

$$\frac{d^2\varphi}{dt^2} = -\omega^2\varphi.$$

A *LINEAR* differential equation of the second order.

General solution: $\varphi(t) = c_1\cos\omega t + c_2\sin\omega t$, $c_1, c_2 \in \mathbb{R}$.

Exact solutions without approximation "$\sin\varphi \approx \varphi$".

(eq. of motion) $\times\dot{\varphi}$:

$$\begin{aligned} \frac{d^2\varphi}{dt^2}\frac{d\varphi}{dt} &= -\omega^2 \sin\varphi \frac{d\varphi}{dt}, \\ \frac{1}{2}\frac{d}{dt}\left(\frac{d\varphi}{dt}\right)^2 &= \frac{d}{dt}(\omega^2\cos\varphi) \\ \xrightarrow{\int(\cdots)dt} \quad \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 &= \omega^2\cos\varphi + (\text{const.}), \\ \tilde{E} := \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - \omega^2\cos\varphi &= (\text{const.}), \end{aligned}$$

$=$ conservation law of the energy.

Remark: the total energy of the pendulum is

$$(\text{kinetic energy}) + (\text{potential energy}) = \frac{ml^2}{2}\left(\frac{d\varphi}{dt}\right)^2 - mgl\cos\varphi = ml^2\tilde{E}.$$

$\alpha :=$ maximum amplitude, i.e., if $\varphi(t_0) = \alpha$, $\dot{\varphi}(t_0) = 0$.

$\Longrightarrow \tilde{E} = -\omega^2 \cos\alpha$, hence

$$\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 = \omega^2(\cos\varphi - \cos\alpha) = 2\omega^2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}\right),$$

$$\frac{d\varphi}{dt} = 2\omega\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}}.$$

Note: $\left|\sin\frac{\varphi}{2}\right| < \sin\frac{\alpha}{2}$, because $|\varphi| \le \alpha < \pi$.

Notations: $k := \sin\frac{\alpha}{2}$, $\theta := \arcsin\left(k^{-1}\sin\frac{\varphi}{2}\right)$, i.e., $\sin\frac{\varphi}{2} = k\sin\theta$.

The equation of motion becomes

$$\frac{d\varphi}{dt} = 2\omega\sqrt{k^2 - k^2\sin^2\theta} = 2k\omega\cos\theta.$$

On the other hand, $\dfrac{d\varphi}{dt} = \dfrac{d\varphi}{d\theta}\dfrac{d\theta}{dt}$ and

$$\frac{d}{d\theta}k\sin\theta = \frac{d}{d\theta}\left(\sin\frac{\varphi}{2}\right),\text{ i.e., } k\cos\theta = \frac{\cos\frac{\varphi}{2}}{2}\frac{d\varphi}{d\theta} = \frac{\sqrt{1-k^2\sin^2\theta}}{2}\frac{d\varphi}{d\theta}$$

Hence,

$$\frac{2k\cos\theta}{\sqrt{1-k^2\sin^2\theta}}\frac{d\theta}{dt} = 2k\omega\cos\theta,$$
$$\frac{1}{\sqrt{1-k^2\sin^2\theta}}\frac{d\theta}{dt} = \omega,$$
$$\int_{\theta(0)}^{\theta(t)}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^t \omega\,dt = \omega t.$$

(Normalisation: $\theta(0) = 0$, i.e., $\varphi(0) = 0$.)

$$F(k,\varphi) := \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

— incomplete *elliptic integral* of the first kind.

$$K(k) := F\left(k,\frac{\pi}{2}\right) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

— complete *elliptic integral* of the first kind.

$\Longrightarrow$ The motion of the pendulum:

$$F\left(\sin\frac{\alpha}{2},\theta(t)\right) = \omega\, t, \text{ or } t(\theta) = \sqrt{\frac{l}{g}} F\left(\sin\frac{\alpha}{2},\theta\right).$$

Therefore the period depends on the amplitude!

Period $= 4 \times$ (time between $\varphi = 0$ and $\varphi = \alpha$)

$$= 4 \times (\text{time between } \theta = 0 \text{ and } \theta = \frac{\pi}{2}) = 4\sqrt{\frac{l}{g}} K\left(\sin\frac{\alpha}{2}\right).$$

- Another expression of the elliptic integral of the first kind

Change the integration variable from ψ to $z := \sin\psi$:

$$dz = \cos\psi\, d\psi = \sqrt{1-z^2}\, d\psi$$

$$\Longrightarrow F(k,\varphi) = \int_0^{\sin\varphi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$