Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$\text{(your final mark)} = \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points}}, 10\right\}$$

- About fifteen problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **1** – **3**: 21 September 2021.

1. (1 pt.) Express the arc length from $(0,0)$ to $(x_0, b\sin\frac{x_0}{a})$ $(x_0 > 0)$ of the graph of the sine curve $y = b\sin\frac{x}{a}$ $(a, b > 0)$ in terms of the elliptic integral of the second kind. Which arc corresponds to the complete elliptic integral $E(k)$?

2. We already know that the arc length of an ellipse is expressed in terms of elliptic integrals. How about the other conics? The answer is as follows.

(i) (1 pt.) Show that the arc length of the hyperbola $(x, y) = (a\cosh t, b\sinh t)$ from $t = 0$ to $t = t_0 > 0$ is formally expressed by an elliptic integral of the second kind as $-ib\,E\left(\frac{\sqrt{a^2+b^2}}{b}, it_0\right)$. ($i = \sqrt{-1}$; There is a formula without a complex number, but it is messy.)

(ii) (1 pt.) Find the formula for arc length of the parabola $y = ax^2$ from $x = 0$ to $x = x_0 > 0$. The result is an elementary function.

3. (1 pt.) If a simple pendulum is made of a stick of length l with negligibly small mass, then it can rotate around the centre. In this case the angle φ is a monotonically increasing function of the time t. (*Not a periodic function!*)

Express its period, namely, the time from $\varphi = 0$ till $\varphi = 2\pi$, in terms of elliptic integrals and the total energy E. (Hint: In the lecture we used a constant $\tilde{E}$ which is equal to E/ml^2. Although there is no "maximum amplitude" α for a rotating "pendulum", we can still use $\tilde{E}$, which plays the role of $-\omega^2\cos\alpha$ in the lecture. Use $k_0 := \sqrt{\frac{2\omega^2}{\omega^2 + \tilde{E}}}$ as the modulus of the elliptic integral. The modulus k used in the lecture is equal to k_0^{-1}.)