Elliptic Integrals and Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:
$$\text{(your final mark)} = \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points}}, 10\right\}$$
- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **4** – **6**: 28 September 2021.

4. (1 pt.) Let $R(x,s)$ be a rational function of x and s and $\varphi(x)$ be a polynomial of degree four without multiple roots. Show that the elliptic integral $\displaystyle\int R(x, \sqrt{\varphi(x)})\,dx$ is rewritten in the form $\displaystyle\int \tilde{R}(y, \sqrt{\psi(y)})\,dy$, where $\tilde{R}(y,s')$ is a rational function in (y,s') and $\psi(y)$ is a polynomial of degree three. (Hint: Use a fractional linear transformation of the variable $x \mapsto y$, which sends one of the roots of $\varphi(x)$ to ∞.)

5. (1 pt.) In the lecture, we claimed that an elliptic integral $\displaystyle\int R(x, \sqrt{\varphi(x)})\,dx$ with $\varphi(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ can be transformed into $\displaystyle\int \tilde{R}(y, \sqrt{\varphi_k(y)})\,dy$ with $\varphi_k(y) = (1-y^2)(1-k^2y^2)$ ($k \neq 0, \pm 1$) by a fractional linear transformation.

(i) Assuming that such a fractional linear transformation exists, express k in terms of the cross-ratio (the anharmonic ratio) $\lambda = \dfrac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}$ of $\alpha_1, \dots, \alpha_4$. (The answer is not unique, but you have only to find one. Hint: show and use the fact that the fractional linear transformation preserves the anharmonic ratio.)

(ii) Show that such a linear transformation really exists.

6. (1 pt.) Reduce the elliptic integral $\displaystyle\int \frac{x^4\,dx}{\sqrt{(1-x^2)(1-2x^2)}}$ to the standard form (a linear combination of an elementary function, the elliptic integrals of the first/second/third kinds).