

Elliptic Functions

Classification of elliptic integrals

§1.3 Classification of elliptic integrals

Recall: Indefinite integrals of the form

$$\int R(x) dx, \quad R(x) : \text{rational function}$$

can be expressed in terms of elementary functions:

rational functions, log, arctan .

Indefinite integrals of the form

$$\int R(x, \sqrt{\varphi(x)}) dx,$$

$R(x, s)$: rational function, $\varphi(x)$: quadratic polynomial

can be reduced to an integral of rational functions, using trigonometric functions and their inverse.

Examples:

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{d \sin \theta}{\cos \theta} && (x = \sin \theta) \\ &= \int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta = \theta = \arcsin x.\end{aligned}$$

$$\begin{aligned}\int \sqrt{1+x^2} dx &= 2 \int \frac{(1+t^2)^2}{(1-t^2)^2} dt && \left(x = \frac{2t}{1-t^2}\right) \\ &= \dots \\ &= \frac{1}{2} \left(x \sqrt{1+x^2} + \log(x + \sqrt{1+x^2})\right).\end{aligned}$$

If $\deg \varphi(x) \geq 3$, in general, integrals $\int R(x, \sqrt{\varphi(x)}) dx$ cannot be expressed in terms of elementary functions.

Note: for simplicity, we assume coefficients of $R(x, s)$ and $\varphi(x)$ are in \mathbb{C} . For example,

$$\arctan x = \int \frac{dx}{1+x^2} = \frac{1}{2i} \int \left(\frac{1}{x-i} - \frac{1}{x+i} \right) dx = \frac{\log(x-i) - \log(x+i)}{2i}.$$

\implies In \mathbb{C} , we do not need \arctan .

Definition:

$\varphi(x)$: polynomial of degree 3 or 4 without multiple roots,

$R(x, s)$: rational function,

\implies integrals of the form $\int R(x, \sqrt{\varphi(x)}) dx$: elliptic integrals.

(When $\deg \varphi(x) \geq 5$: *hyperelliptic* integrals.)

The case $\deg \varphi(x) = 3$ and $\deg \varphi(x) = 4$ are essentially the same!

Example: $\deg \varphi(x) = 3$.

$$\varphi(x) = a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3),$$

$\alpha_1, \alpha_2, \alpha_3$: distinct by assumption.

Take a fractional linear transformation: $x = T(y) = \frac{Ay + B}{Cy + D}$, such that

- $C + D = 0$, i.e., $T(1) = \infty$.
- $T(\infty) = A/C \neq \alpha_i$ ($i = 1, 2, 3$), i.e., $\beta_i := T^{-1}(\alpha_i) \neq \infty$.

$$\begin{aligned} \implies x - \alpha_i &= \frac{Ay + B}{Cy + D} - \frac{A\beta_i + B}{C\beta_i + D} \\ &= \text{const.} \times \frac{y - \beta_i}{y - 1}. \quad \left(\text{const.} = \frac{A - \alpha_i C}{C} \right) \end{aligned}$$

$$\begin{aligned}
R(x, \sqrt{\varphi(x)}) &= R \left(\frac{Ay + B}{Cy + D}, \text{const.} \sqrt{\frac{(y - \beta_1)(y - \beta_2)(y - \beta_3)}{(y - 1)^3}} \right) \\
&= R \left(\frac{Ay + B}{Cy + D}, \text{const.} \frac{\sqrt{(y - \beta_1)(y - \beta_2)(y - \beta_3)(y - 1)}}{(y - 1)^2} \right) \\
&= \tilde{R}(y, \sqrt{(y - \beta_1)(y - \beta_2)(y - \beta_3)(y - 1)}).
\end{aligned}$$

($\tilde{R}(y, t)$: new rational function) and

$$dx = \frac{dx}{dy} dy = \frac{AD - BC}{(Cy + D)^2} dy.$$

$$\implies \int R(x, \sqrt{\varphi(x)}) dx = \int \tilde{\tilde{R}}(y, \sqrt{\psi(y)}) dy,$$

$$\psi(y) = (y - \beta_1)(y - \beta_2)(y - \beta_3)(y - 1), \quad \tilde{\tilde{R}}(y, t) = \tilde{R}(y, t) \times \frac{AD - BC}{(Cy + D)^2}.$$

Exercise: Rewrite

$$\int R(x, \sqrt{\varphi(x)}) dx \quad (\deg \varphi = 4) \rightsquigarrow \int R'(y, \sqrt{\psi(y)}) dy \quad (\deg \psi = 3).$$

Hereafter $\deg \varphi(x) = 4$.

Note:

- $\varphi(x) = \prod_{i=1}^4 (x - \alpha_i)$ has four parameters $(\alpha_1, \dots, \alpha_4)$.

- fractional linear transformations $\frac{Ax + B}{Cx + D}$ ($AD - BC = 1$) determined by three parameters.

\implies remains four – three = one parameter.

In fact, may assume

$$\varphi(x) = \varphi_k(x) = (1 - x^2)(1 - k^2x^2),$$

by using a fractional linear transformation T , such that

$$(T\alpha_1, T\alpha_2, T\alpha_3, T\alpha_4) = (1, k^{-1}, -1, -k^{-1}).$$

Exercise:

(i) Express k in terms of the cross ratio $\lambda = \frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}$ of $(\alpha_1, \dots, \alpha_4)$.

(ii) Show that such T exists.

(iii) $\int R(x, \sqrt{\varphi(x)}) dx \rightsquigarrow \int R_k(y, \sqrt{(1 - y^2)(1 - k^2y^2)}) dy$.

Theorem (Legendre-Jacobi standard forms):

Any elliptic integral is a linear combination of

- elementary functions (combinations of rational functions, log, inverse trigonometric functions and $\sqrt{\varphi(x)}$),

- $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ (*elliptic integral of the first kind*),

- $\int \sqrt{\frac{1-k^2x^2}{1-x^2}} dx = \int \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}} dx$
(*elliptic integral of the second kind*),

- $\int \frac{dx}{(x^2-\alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}$, α : parameter
(*elliptic integral of the third kind*). □

k : modulus of the elliptic integral.

Proof:

Suppose an elliptic integral $\int R(x, \sqrt{\varphi(x)}) dx$ is given.

$s := \sqrt{\varphi(x)}$, i.e., $s^2 = \varphi(x)$.

$R(x, s)$: rational function $\implies \exists$ polynomials $P_1(x), P_2(x), Q_1(x), Q_2(x)$,

$$\begin{aligned} R(x, \sqrt{\varphi(x)}) &= \frac{(\text{polynomial of } x, s)}{(\text{polynomial of } x, s)} = \frac{P_1(x) + P_2(x)s}{Q_1(x) + Q_2(x)s} \\ &= \frac{(P_1(x) + P_2(x)s)(Q_1(x) - Q_2(x)s)}{(Q_1(x) + Q_2(x)s)(Q_1(x) - Q_2(x)s)} \\ &= \frac{\tilde{P}_1(x) + \tilde{P}_2(x)s}{Q_1(x)^2 - Q_2(x)^2\varphi(x)} \quad (\tilde{P}_1(x), \tilde{P}_2(x) : \text{polynomials}) \\ &= R_1(x) + \tilde{R}_2(x)s \quad (R_1(x), R_2(x) : \text{rational functions}) \\ &= R_1(x) + \frac{R_2(x)}{s} \quad (R_2(x) = \tilde{R}_2(x)\varphi(x)). \end{aligned}$$

$$\int R(x, \sqrt{\varphi(x)}) dx = \int R_1(x) dx + \int \frac{R_2(x)}{s} dx.$$

Since the rational function $R_2(x)$ is expanded as

$$R_2(x) = (\text{polynomial}) + \sum_{j=1}^M \sum_{n=1}^{n_j} \frac{a_{jn}}{(x - \alpha_j)^n},$$

the integral $\int R(x, \sqrt{\varphi(x)}) dx$ is a linear combination of

- the integral of a rational function $R_1(x)$,
- $I_n := \int \frac{x^n}{s} dx$ ($n = 0, 1, 2, \dots$),
- $J_n(\alpha) := \int \frac{dx}{(x - \alpha)^n s}$ ($n = 0, 1, 2, \dots$).

Recurrence relations:

- recurrence relations among I_n 's: integrate the relation,

$$\begin{aligned}\frac{d}{dx} x^n s &= nx^{n-1} s + \frac{x^n}{2s} \frac{d\varphi}{dx} \\ &= \frac{nx^{n-1}\varphi(x)}{s} + \frac{x^n(\text{polynomial of deg} = 3)}{s} \\ \implies x^n s &= \begin{cases} c_{n+3}I_{n+3} + \cdots + c_n I_n + c_{n-1}I_{n-1}, & (n \neq 0), \\ c_3 I_3 + \cdots + c_0 I_0, & (n = 0). \end{cases}\end{aligned}$$

$\xrightarrow{\text{induction}}$ I_n ($n \geq 3$) = a linear combination of (polynomial) $\times s$, I_2 , I_1 , I_0 .

- recurrence relations among $J_n(\alpha)$'s: For $n \geq 1$,

$$\begin{aligned} \frac{d}{dx} \frac{s}{(x - \alpha)^n} &= \frac{-ns}{(x - \alpha)^{n+1}} + \frac{1}{2(x - \alpha)^n} \frac{d\varphi}{dx} \\ &= \frac{1}{(x - \alpha)^{n+1} s} \left(-n\varphi(x) + \frac{x - \alpha}{2} \frac{d\varphi}{dx} \right) \\ &= \frac{1}{(x - \alpha)^{n+1} s} \sum_{i=0}^4 d_{n,i} (x - \alpha)^i, \end{aligned}$$

where $d_{n,0} = -n\varphi(\alpha)$, $d_{n,1} = \left(\frac{1}{2} - n\right) \varphi'(\alpha)$.

Integrating this relation we have

$$\frac{s}{(x - \alpha)^n} = d_{n,0} J_{n+1} + \cdots + d_{n,4} J_{n-3}.$$

1. If $\varphi(\alpha) \neq 0 \Rightarrow d_{n,0} \neq 0$.

J_{n+1} is a linear combination of J_n, \dots, J_{n-3} and $\frac{s}{(x-\alpha)^n}$.

$\xrightarrow{\text{induction}} J_n (n \geq 2)$ = a linear combination of J_1, J_0, J_{-1}, J_{-2} and (rational function) $\times s$.

2. If $\varphi(\alpha) = 0 \Rightarrow \varphi'(\alpha) \neq 0$. (φ does not have multiple roots.)

Hence $d_{n,0} = 0, d_{n,1} \neq 0$.

J_n is a linear combination of J_{n-1}, \dots, J_{n-3} and $\frac{s}{(x-\alpha)^n}$.

$\xrightarrow{\text{induction}} J_n (n \geq 1)$ = a linear combination of J_0, J_{-1}, J_{-2} and (rational function) $\times s$.

On the other hand, $J_{-2} = \int \frac{(x-\alpha)^2}{s} dx, J_{-1} = \int \frac{x-\alpha}{s} dx$ and

$J_0 = \int \frac{dx}{s}$ are linear combinations of I_0, I_1 and I_2 .

Summarising, in any case, $\int R(x, \sqrt{\varphi(x)}) dx$ is a linear combination of integrals of rational functions, (rational function) $\times \sqrt{\varphi(x)}$ and

$$I_0 = \int \frac{dx}{\sqrt{\varphi(x)}},$$

$$I_1 = \int \frac{x dx}{\sqrt{\varphi(x)}},$$

$$I_2 = \int \frac{x^2 dx}{\sqrt{\varphi(x)}},$$

$$J_1(\alpha) = \int \frac{dx}{(x - \alpha)\sqrt{\varphi(x)}}.$$

Now, recall that we may assume $\varphi(x) = \sqrt{(1 - x^2)(1 - k^2 x^2)}$.

- I_0 : the elliptic integral of the first kind.

- $$I_1 = \frac{1}{2} \int \frac{dt}{\sqrt{(1-t)(1-kt)}} \quad (x = t^2)$$

= elementary function (inverse trigonometric function).

- $$I_2 = \int \frac{\frac{1}{k^2}(1 - (1 - k^2x^2))}{\sqrt{(1-x^2)(1-k^2x^2)}} dx = \frac{1}{k^2}I_0 - \frac{1}{k^2} \int \sqrt{\frac{1-k^2x^2}{1-x^2}} dx.$$

= (elliptic integral of the first kind) + (the second kind).

- $$J_1 = \int \frac{(x + \alpha) dx}{(x^2 - \alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= \frac{1}{2} \int \frac{dt}{(t - \alpha^2)\sqrt{(1-t)(1-k^2t)}} + \alpha \int \frac{dx}{(x^2 - \alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

= (elementary function) + (elliptic integral of the third kind).

□

Another standard forms (Riemann standard form):

$$\int \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}},$$

$$\int \frac{x dx}{\sqrt{x(1-x)(1-\lambda x)}},$$

$$\int \frac{dx}{(x-\alpha)\sqrt{x(1-x)(1-\lambda x)}}.$$