Elliptic Integrals and Elliptic Functions

Takashi Takebe

28 September 2021

- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$(\text{your final mark}) = \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points}}, 10\right\}$$

- About fifteen problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **7** – **8**: 5 October 2021.

7. (1 pt.) Prove that,

$$\begin{aligned} K(k) &\to \infty, \\ \operatorname{sn}(u,k) &\to \tanh u = \frac{\sinh u}{\cosh u}, \\ \operatorname{cn}(u,k),\ \operatorname{dn}(u,k) &\to \operatorname{sech} u = \frac{1}{\cosh u}, \end{aligned}$$

when the modulus $k \in (0,1)$ tends to 1.

8. (1 pt.) Complete the proof of the addition formula of $\operatorname{sn} u$, finishing the computation omitted in the lecture, especially $\dfrac{dN}{du}D = N\dfrac{dD}{du}$. Then, using this addition formula, prove the addition formulae for $\operatorname{cn} u$ and $\operatorname{dn} u$:

$$\begin{aligned} \operatorname{cn}(u+v) &= \frac{\operatorname{cn} u\ \operatorname{cn} v - \operatorname{sn} u\ \operatorname{sn} v\ \operatorname{dn} u\ \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u\ \operatorname{sn}^2 v}, \\ \operatorname{dn}(u+v) &= \frac{\operatorname{dn} u\ \operatorname{dn} v - k^2 \operatorname{sn} u\ \operatorname{sn} v\ \operatorname{cn} u\ \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u\ \operatorname{sn}^2 v}. \end{aligned}$$

It is enough to check the consistency of these formulae with the definitions of cn and dn. (You can omit checking the signs of square roots.)

(Hint: The following formulae might be useful.
For $\operatorname{cn}(u+v)$: $1 - k^2 \operatorname{sn}^2 u\ \operatorname{sn}^2 v = \operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 v = \operatorname{cn}^2 v + \operatorname{sn}^2 v \operatorname{dn}^2 u$.
For $\operatorname{dn}(u+v)$: $1 - k^2 \operatorname{sn}^2 u\ \operatorname{sn}^2 v = \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u \operatorname{cn}^2 v = \operatorname{dn}^2 v + k^2 \operatorname{sn}^2 v \operatorname{cn}$.)