Elliptic Functions

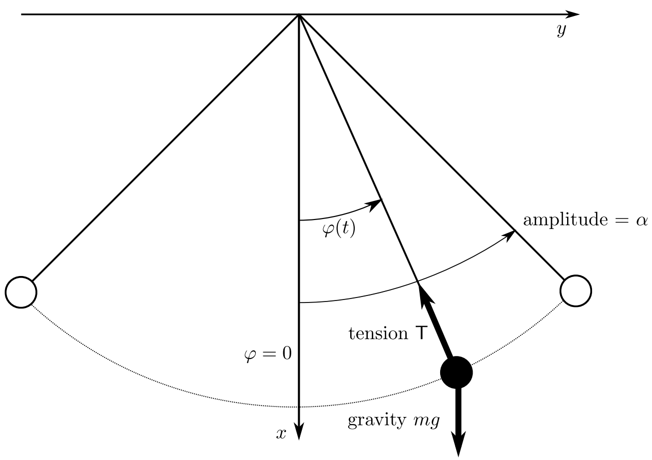
Jacobi's elliptic functions over $\mathbb{R}$

§3.1 Jacobi's elliptic functions

Recall: the motion of a simple pendulum is described by

$$t(\theta) = \sqrt{\frac{l}{g}} F\left(\sin\frac{\alpha}{2}, \theta\right) = \sqrt{\frac{l}{g}} \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where $\alpha =$max. amplitude, $k = \sin\frac{\alpha}{2}$, $\sin\theta = k^{-1}\sin\frac{\varphi}{2}$.



Better description of motion: "amplitude $\varphi =$ function of time t".

$\Longrightarrow$ Consider the *inverse function*!

Assume $0 \le k < 1$.

Definition:

Jacobi's elliptic function $\mathrm{sn}(u) = \mathrm{sn}(u,k) :=$ the inverse function of

$$u(x) = u(x,k) := \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

on $-1 \le x \le 1$ and, consequently,

$-K(k) \le u \le K(k)$ (= the complete elliptic integral of the first kind.)

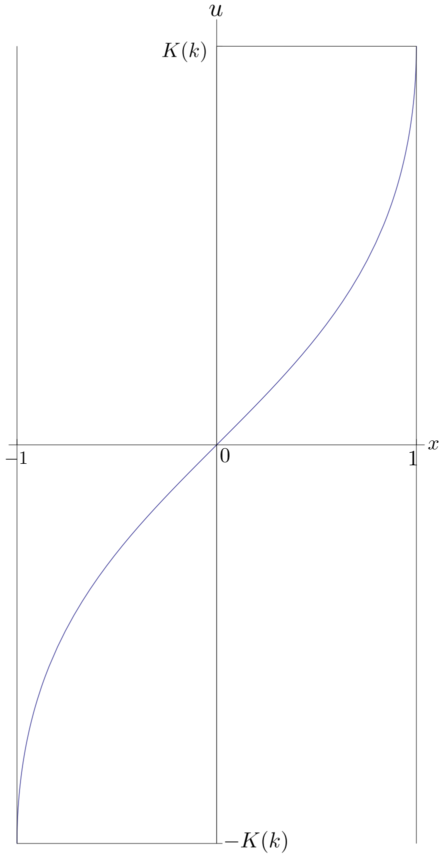
$\mathrm{sn}(u,k) =$ analogue of $\sin u$: in fact, when $k=0$,

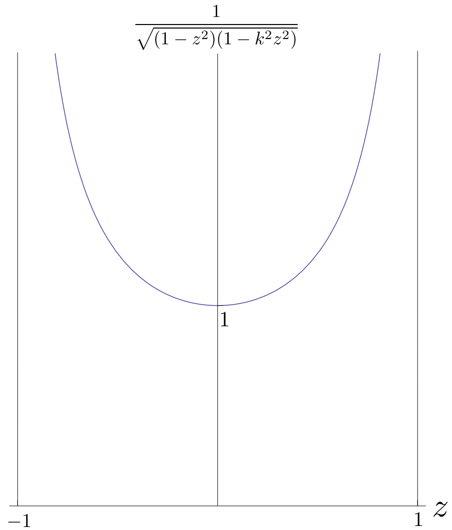
$$\mathrm{sn}(u,k) = \text{the inverse function of } \left(u(x) = \int_0^x \frac{dz}{\sqrt{(1-z^2)}} = \arcsin x \right).$$

Namely,

$$\mathrm{sn}(u,0) = \sin u, \quad K(0) = \frac{\pi}{2}.$$

Graphs of $u(x,k)$ and its integrand $\dfrac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$.





$$u(x,k) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \xrightarrow{k \longrightarrow 0} \int_0^x \frac{dz}{\sqrt{(1-z^2)}} = \arcsin x$$

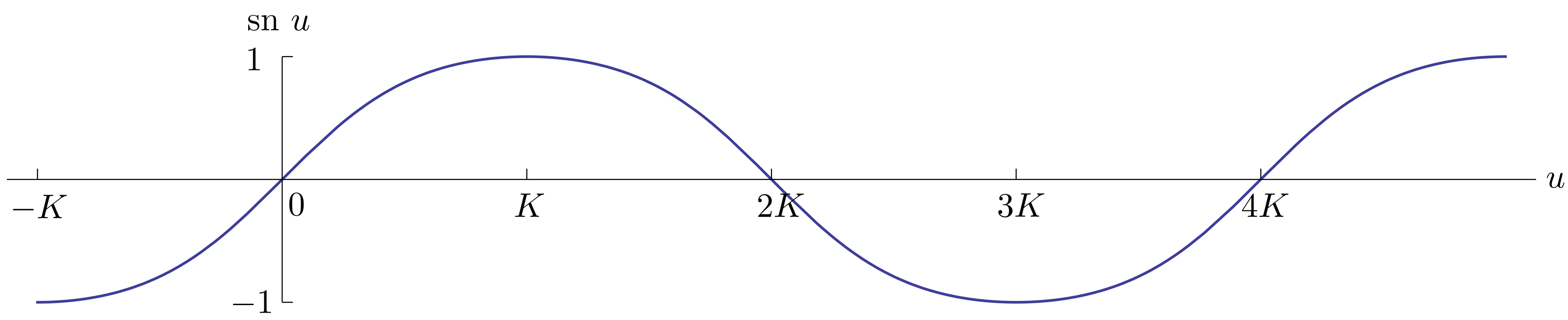
Important property of $\sin$ = periodicity: $\sin(u + 2\pi) = \sin u$.

$\Longrightarrow$ Extend sn to $\mathbb{R}$ by periodicity:

$$\mathrm{sn}(u + 2K(k), k) = -\,\mathrm{sn}(u, k), \qquad \mathrm{sn}(u + 4K(k), k) = \mathrm{sn}(u, k).$$

(Justification given by add. formula or in § "Complex elliptic functions".)

Graph of sn



The motion of the simple pendulum revisited:

$$\begin{aligned} t(\theta) &= \sqrt{\frac{l}{g}} \int_0^{\theta} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad \left(k = \sin\frac{\alpha}{2}\right) \\ &= \sqrt{\frac{l}{g}} \int_0^{\sin\theta} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \qquad (z = \sin\phi) \\ &= \sqrt{\frac{l}{g}} \int_0^{k^{-1}\sin\frac{\varphi}{2}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \end{aligned}$$

Using Jacobi's sn function,

$$\sin\frac{\varphi(t)}{2} = k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}t, k\right), \qquad \varphi(t) = 2\arcsin\left(k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}t, k\right)\right).$$

§3.2 Properties of Jacobi's elliptic functions

Introduce cn (analogue of $\cos$) and dn:

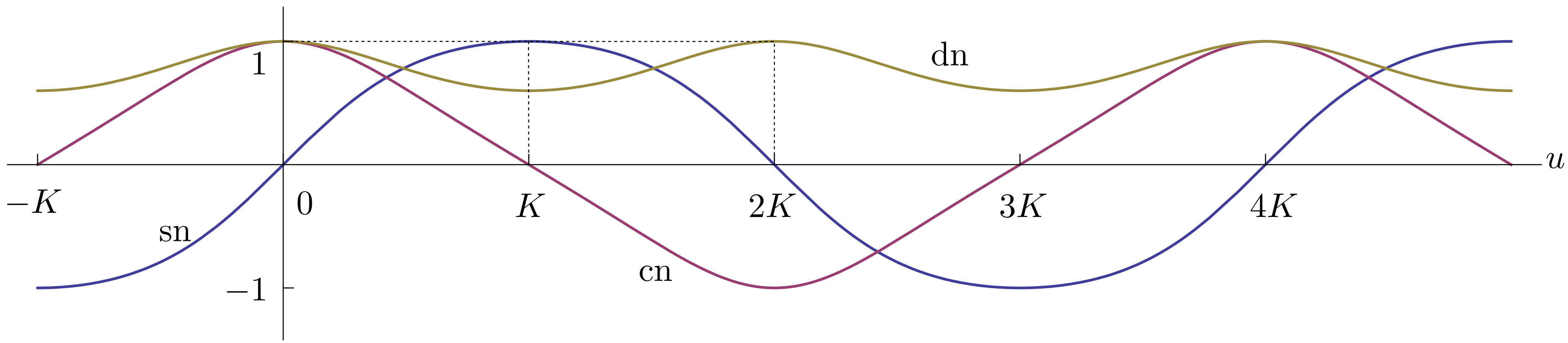
$$\mathrm{cn}(u) = \mathrm{cn}(u,k) := \sqrt{1-\mathrm{sn}^2(u,k)}, \qquad (\mathrm{cn}(0)=1),$$

$$\mathrm{dn}(u) = \mathrm{dn}(u,k) := \sqrt{1-k^2\,\mathrm{sn}^2(u,k)}, \qquad (\mathrm{dn}(0)=1),$$

and extend by periodicity.

$k \to 0$: $K(k) \to \pi$, $\mathrm{sn}\,u \to \sin u$, $\mathrm{cn}\,u \to \cos u$, $\mathrm{dn}\,u \to 1$.

Graphs of sn, cn and dn:



Exercise: Show that, when $k \to 1$,

$$K(k) \to \infty,$$

$$\operatorname{sn}(u,k) \to \tanh u = \frac{\sinh u}{\cosh u},$$

$$\operatorname{cn}(u,k), \operatorname{dn}(u,k) \to \operatorname{sech} u = \frac{1}{\cosh u}.$$

$\operatorname{sn} u =$ the inverse function of elliptic integral $F(k,x)$, i.e.,

$$u = F(k, \operatorname{sn} u) = \int_0^{\operatorname{sn} u} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

$$\frac{d}{du}\operatorname{sn} u = \frac{1}{\frac{\partial}{\partial x}F(k,x)\Big|_{x=\operatorname{sn} u}} = \frac{1}{\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\Big|_{x=\operatorname{sn} u}}$$

$$= \sqrt{1-\operatorname{sn}^2 u}\sqrt{1-k^2\operatorname{sn}^2 u} = \operatorname{cn} u \ \operatorname{dn} u.$$

As corollaries,

$$\frac{d}{du}\operatorname{cn} u = \frac{d}{du}\sqrt{1-\operatorname{sn}^2 u} = \frac{-\operatorname{sn} u \frac{d\operatorname{sn} u}{du}}{\sqrt{1-\operatorname{sn}^2 u}} = -\operatorname{sn} u \ \operatorname{dn} u.$$

$$\frac{d}{du}\operatorname{dn} u = \frac{d}{du}\sqrt{1-k^2\operatorname{sn}^2 u} = \frac{-k^2\operatorname{sn} u \frac{d\operatorname{sn} u}{du}}{\sqrt{1-k^2\operatorname{sn}^2 u}} = -k^2\operatorname{sn} u \ \operatorname{cn} u.$$

Summarising,

$$
\frac{d\,\mathrm{sn}\,u}{du} = \mathrm{cn}\,u\ \mathrm{dn}\,u, \qquad \xrightarrow{k\to 0} \qquad \frac{d\sin u}{du} = \cos u,
$$

$$
\frac{d\,\mathrm{cn}\,u}{du} = -\,\mathrm{sn}\,u\ \mathrm{dn}\,u, \qquad \xrightarrow{k\to 0} \qquad \frac{d\cos u}{du} = -\sin u,
$$

$$
\frac{d\,\mathrm{dn}\,u}{du} = -k^2\,\mathrm{sn}\,u\ \mathrm{cn}\,u.
$$

Addition formulae:

Addition formula of $\sin$: $\sin(u+v) = \sin u\ \cos v + \cos u\ \sin v$.

Addition formula of $\tanh$: $\tanh(u+v) = \dfrac{\tanh u + \tanh v}{1 + \tanh u\ \tanh v}$.

$\mathrm{sn}(u,k)$ interpolates $\sin$ ($k=0$) and $\tanh$ ($k=1$).

$\Longrightarrow$ A natural guess is "sn has an addition formula."

Let us "interpolate" the above formulae!

Addition formula of $\sin$ without $\cos$:

$$\sin(u+v) = \sin u \frac{d \sin v}{dv} + \frac{d \sin u}{du} \sin v.$$

Note $\dfrac{d\tanh u}{du} = 1 - \tanh^2 u$. Hence

$$\tanh u \frac{d\tanh v}{dv} + \frac{d\tanh u}{du}\tanh v = (\tanh u + \tanh v)(1 - \tanh u \ \tanh v).$$

Addition formula of $\tanh$ can be rewritten as

$$\tanh(u+v) = \frac{\tanh u \frac{d\tanh v}{dv} + \frac{d\tanh u}{du}\tanh v}{1 - \tanh^2 u \ \tanh^2 v}.$$

A possible interpolation of the addition formulae of $\sin$ and $\tanh$:

$$\begin{aligned}\mathrm{sn}(u+v) &= \frac{\mathrm{sn}\, u \frac{d\,\mathrm{sn}\, v}{dv} + \frac{d\,\mathrm{sn}\, u}{du}\mathrm{sn}\, v}{1 - k^2\,\mathrm{sn}^2 u \ \mathrm{sn}^2 v} \\ &= \frac{\mathrm{sn}\, u \ \mathrm{cn}\, v \ \mathrm{dn}\, v + \mathrm{sn}\, v \ \mathrm{cn}\, u \ \mathrm{dn}\, u}{1 - k^2\,\mathrm{sn}^2 u \ \mathrm{sn}^2 v}.\end{aligned}$$

In fact this is true!

<u>Proof</u>: $u + v \to c$, $v \to c - u$,

$$F(u) := \frac{\operatorname{sn} u \ \operatorname{cn}(c-u) \ \operatorname{dn}(c-u) + \operatorname{sn}(c-u) \ \operatorname{cn} u \ \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \ \operatorname{sn}^2(c-u)}.$$

<u>Claim</u>: $\dfrac{dF}{du} = 0$, when c is fixed.

Claim $\Longrightarrow F(0) = F(u)$ and, since $\operatorname{sn} 0 = 0$, $\operatorname{cn} 0 = \operatorname{dn} 0 = 1$,

$$\operatorname{sn} c = \frac{\operatorname{sn} u \ \operatorname{cn}(c-u) \ \operatorname{dn}(c-u) + \operatorname{sn}(c-u) \ \operatorname{cn} u \ \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \ \operatorname{sn}^2(c-u)}.$$

Substitutig $c = u + v$, we obtain the addition formula. □

<u>Proof of the claim</u>:

$N :=$ numerator of $F(u) = \operatorname{sn} u \ \operatorname{cn}(c-u) \ \operatorname{dn}(c-u) + \operatorname{sn}(c-u) \ \operatorname{cn} u \ \operatorname{dn} u$

$D :=$ denominator of $F(u) = 1 - k^2 \operatorname{sn}^2 u \ \operatorname{sn}^2(c-u)$.

Long computation (Exercise!) shows

$$\frac{dN}{du}D = N\frac{dD}{du}.$$

Therefore

$$\frac{dF}{du} = \frac{\frac{dN}{du}D - N\frac{dD}{du}}{D^2} = 0.$$

□

Addition formulae of cn and dn:

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\,u\ \mathrm{cn}\,v - \mathrm{sn}\,u\ \mathrm{sn}\,v\ \mathrm{dn}\,u\ \mathrm{dn}\,v}{1 - k^2\,\mathrm{sn}^2\,u\ \mathrm{sn}^2\,v},$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\,u\ \mathrm{dn}\,v - k^2\,\mathrm{sn}\,u\ \mathrm{sn}\,v\ \mathrm{cn}\,u\ \mathrm{cn}\,v}{1 - k^2\,\mathrm{sn}^2\,u\ \mathrm{sn}^2\,v}.$$