Elliptic Functions

Riemann surfaces of algebraic functions.

§4.1 Riemann surface of algebraic functions.

Hitherto: elliptic integrals and elliptic functions (mainly) over $\mathbb{R}$.

Let us *complexify* the theories!

Want: integrals of $R(x, \sqrt{\varphi(x)})$ on $\mathbb{C}$.

$\longrightarrow$ A problem of multi-valuedness (branches) of $\sqrt{\varphi(x)}$ occurs.

The simplest case: $\sqrt{z}$.

What is $\sqrt{z}$*?* — "w which satisfies $w^2 = z$".

Then $\sqrt{z}$ cannot be uniquely determined: if $w^2 = z$, then $(-w)^2 = z$.

Where does this "$-$" sign come from?

$z = re^{i\theta}$ ($r = |z|$, $\theta = \arg z$; polar form) $\Longrightarrow \sqrt{z} = \sqrt{r}e^{i\theta/2}$.

- For $r \in \mathbb{R}_{>0}$, $\sqrt{r} > 0$ is uniquely determined.
- $\theta = \arg z$ is NOT unique! $\arg z$ is determined only up to $2\pi\mathbb{Z}$:

$$z = re^{i\theta} = re^{i(\theta \pm 2\pi)} = re^{i(\theta \pm 4\pi)} = \cdots = re^{i(\theta + 2n\pi)}.$$

Correspondingly,

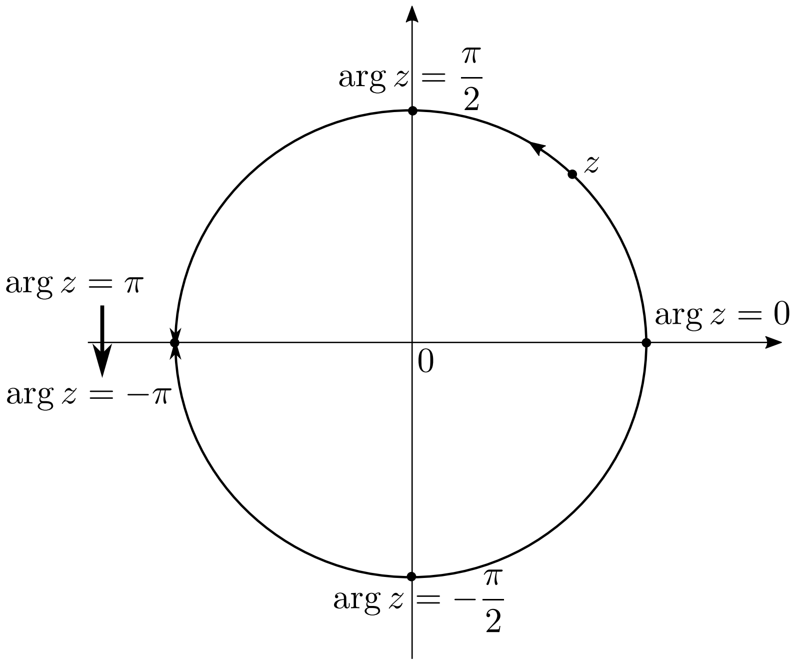
$$\sqrt{z} = \sqrt{r}e^{i(\theta+2n\pi)/2} = \sqrt{r}e^{i\theta+in\pi} = (-1)^n\sqrt{r}e^{i\theta}.$$

Two solutions to the multi-valuedness problem:

1. Restrict the range of $\arg$ (e.g., $-\pi < \arg z \leqq \pi$).

   — Not convenient, for example, to consider $\sqrt{z}$ on a curve around 0.

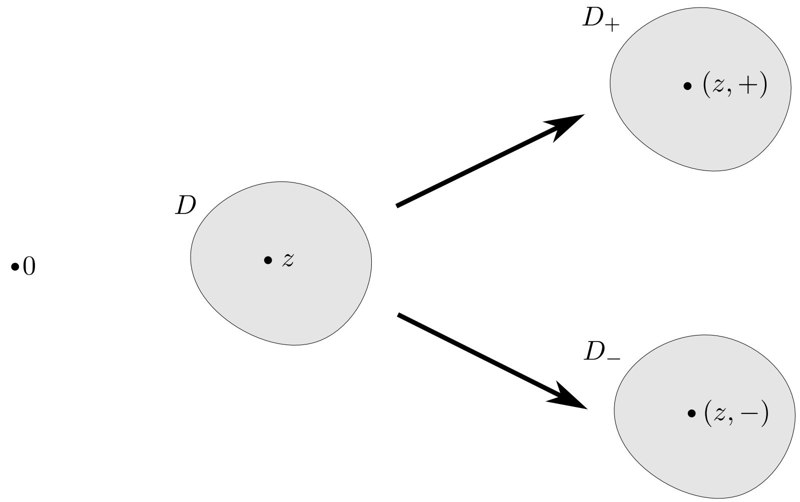
   (cf. Figure.) The range is arbitrarily chosen.



2. Double the domain of definition (Riemann's idea):

   Assign two "points" $(z, +)$ and $(z, -)$ to each $z \neq 0$.

D: "small" domain, $0 \notin D$. $\Longrightarrow$ D splits to D_+ and D_-.



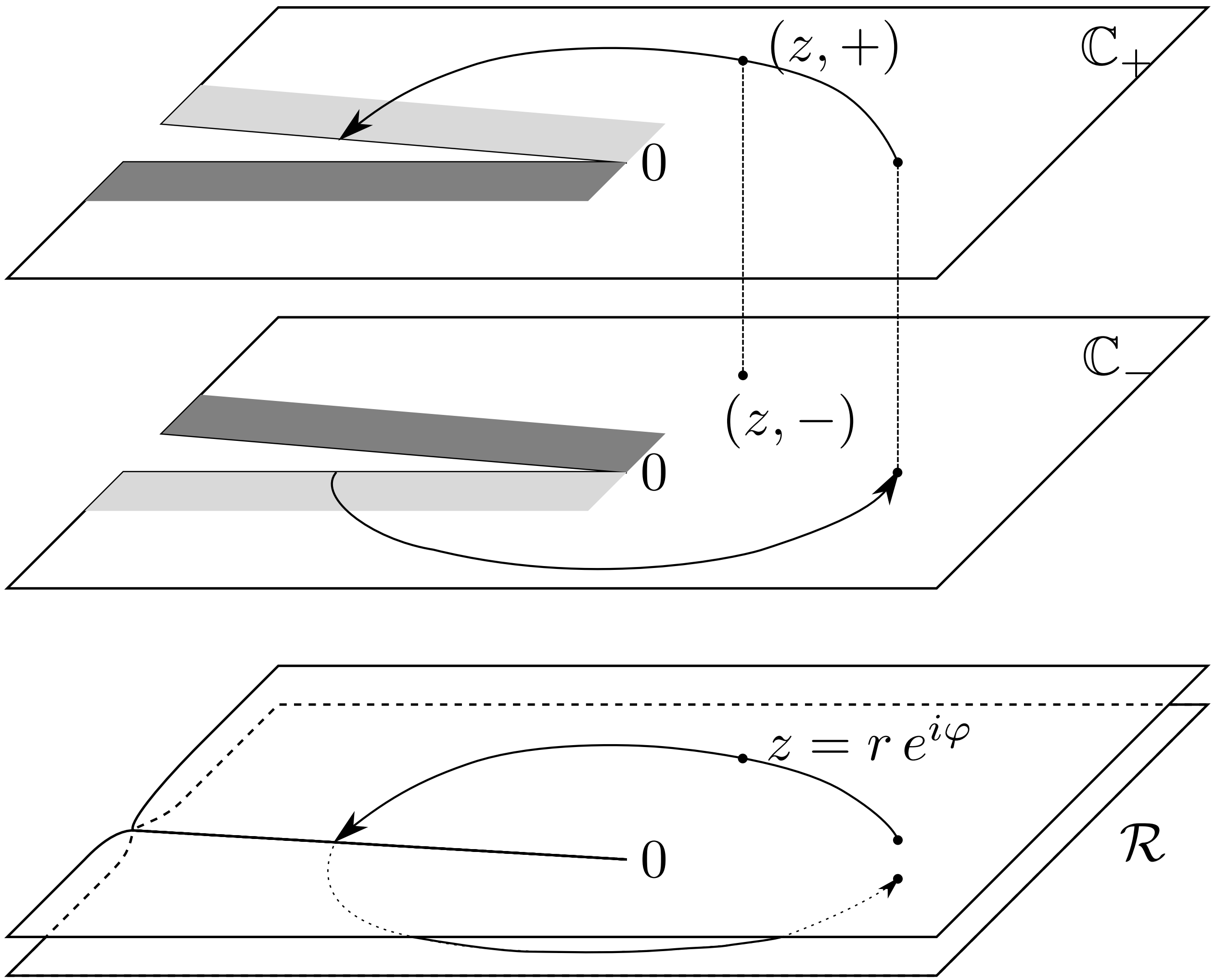
$$z = re^{i\theta} \ (\theta \in (-\pi, \pi]) \longrightarrow \begin{cases} \sqrt{(z,+)} = +\sqrt{r}e^{i\theta/2}, \\ \sqrt{(z,-)} = -\sqrt{r}e^{i\theta/2}. \end{cases}$$

How about $z = 0$? Since $\sqrt{0} = 0$ is unique, it should not be split.

Then what occurs with the whole plane $\mathbb{C}$?

Answer (by Riemann):

Glue $(\mathbb{C} \setminus \{0\})_+$ & $(\mathbb{C} \setminus \{0\})_-$ (= two copies of $\mathbb{C} \setminus \{0\}$) as follows:



Motion of $z = re^{i\varphi}$ ($r > 0$, $\varphi \in [0, 2\pi]$):

1. When $\varphi \leqq \pi$, z moves on the upper plane.
2. When φ exceeds π, z transfers to the lower plane.
3. When $\varphi = 2\pi$, z does not come back to the start!

$$\varphi = 0 \leftrightarrow (z, +) \rightsquigarrow (z, -) \leftrightarrow \varphi = 2\pi$$

Correspondingly, when $z = re^{i(\varphi+\theta)}$ ($0 \leqq \theta \leqq 2\pi$) moves arround 0:

$$\sqrt{z} = \sqrt{r}e^{i\varphi/2} \xrightarrow{0 \leqq \theta \leqq 2\pi} \sqrt{z} = -\sqrt{r}e^{i\varphi/2}.$$

Summarising: $\sqrt{z}$ should be defined on

$$\begin{array}{lccccc} \mathcal{R} := & (\mathbb{C} \setminus \{0\})_+ & \cup & \{0\} & \cup & (\mathbb{C} \setminus \{0\})_- \\ \sqrt{z}: & \sqrt{r}e^{i\varphi/2} & & 0 & & -\sqrt{r}e^{i\varphi/2} \end{array}$$

This $\mathcal{R}$ is the *Riemann surface* of $\sqrt{z}$. ... quite "hand-made".

• <u>Systematic construction of the Riemann surface</u>:

Points of $\mathcal{R}$: $(z, \pm) \rightsquigarrow (z, w = \pm\sqrt{z} = \pm\sqrt{r}e^{i\varphi/2})$.

$$\mathcal{R} := \{(z, w) \mid F(z, w) := w^2 - z = 0\} \subset \mathbb{C}^2.$$

- 0 is naturally included in $\mathcal{R}$ as $(0, 0)$.
- $\mathcal{R}$ has natural topology as a subset of $\mathbb{C}^2$.
- $\mathcal{R}$ is a *one-dimensional complex manifold*.

• Review: manifold

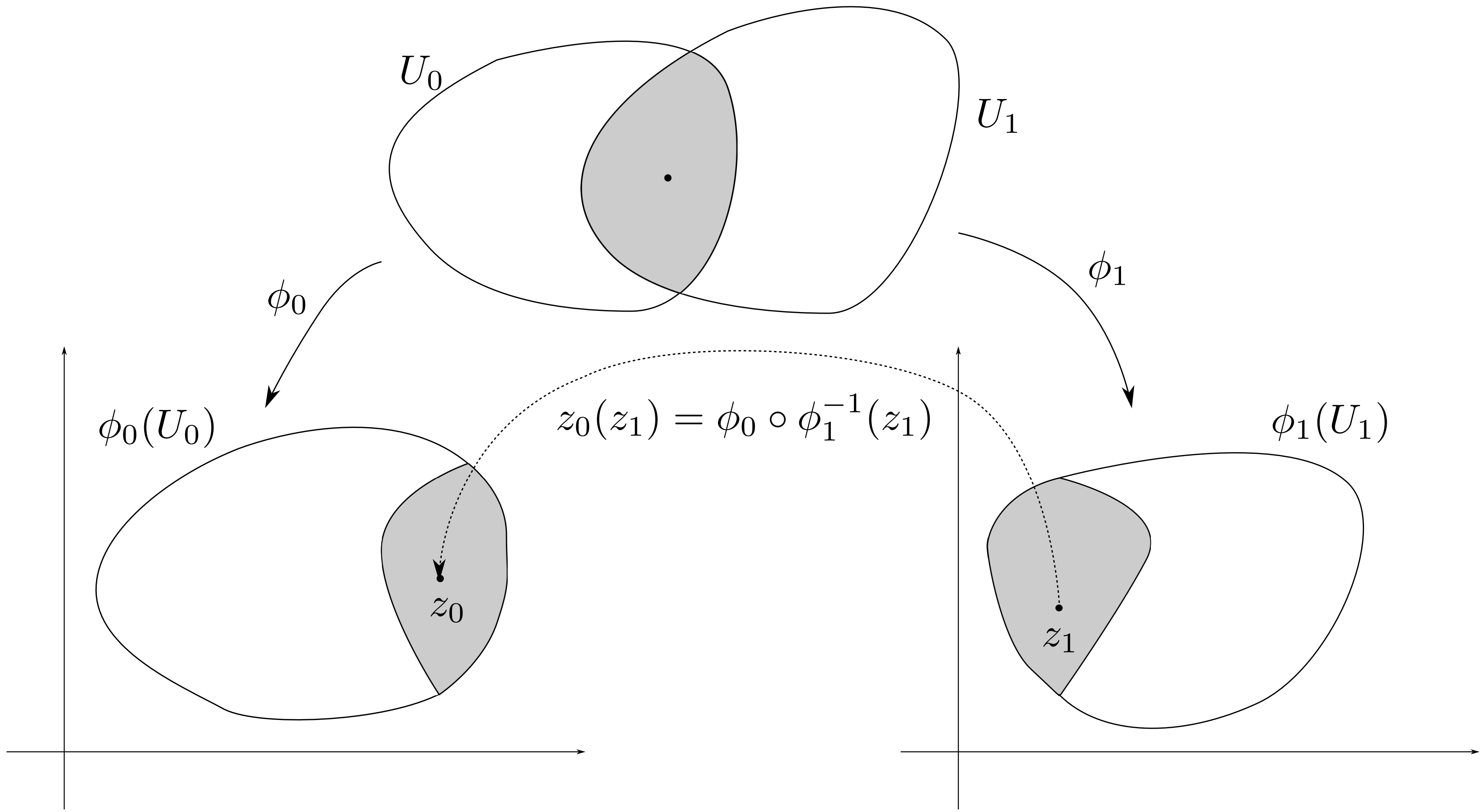
X: real (C^r-)manifold

- X: Hausdorff space.
- $\{(U_\lambda, \phi_\lambda)\}_{\lambda \in \Lambda}$: *atlas* of X, i.e.,

$$U_\lambda \subset X : \text{ open}, \ \bigcup_{\lambda \in \Lambda} U_\lambda = X,$$

$$\phi_\lambda : U_\lambda \to V_\lambda \in \mathbb{R}^N : \text{homeomorphism}$$

- $\phi_\lambda \circ \phi_\mu^{-1} : \phi_\mu(U_\lambda \cap U_\mu) \to \phi_\lambda(U_\lambda \cap U_\mu)$: C^r-diffeomorphism.


U_0
U_1
ϕ_0
ϕ_1
$\phi_0(U_0)$
$z_0(z_1) = \phi_0 \circ \phi_1^{-1}(z_1)$
$\phi_1(U_1)$
z_0
z_1

Complex manifold: $\mathbb{R} \rightsquigarrow \mathbb{C}$, C^r-diffeomorphism $\rightsquigarrow$ holomorphic bijection.

Theorem:

Assumptions:

- $F(z,w)$: polynomial.
- $\left(F, \dfrac{\partial F}{\partial z}, \dfrac{\partial F}{\partial w}\right) \neq (0,0,0)$ on a domain $U \subset \mathbb{C}^2$.

Then $\{(z,w) \mid F(z,w) = 0\} \cap U$ is a one-dimensional complex manifold (possibly non-connected). $\square$

Remark:

May assume that $F(z,w)$ is a holomorphic function in (z,w).

We use only the polynomial case.

Lemma: (Holomorphic implicit function theorem)

$F(z,w)$: as above. Assume $F(z_0,w_0)=0$, $\dfrac{\partial F}{\partial w}(z_0,w_0) \neq 0$.

Then,

- $\exists r, \rho > 0$ such that

  $$\left\{ (z,w) \,\middle|\, \begin{matrix} |z-z_0|<r, |w-w_0|<\rho \\ F(z,w)=0 \end{matrix} \right\} \ni (z,w) \mapsto z \in \{ z \mid |z-z_0| < r \}$$

  is bijective.

- the component $\varphi(z)$ of the inverse map $z \mapsto (z, \varphi(z))$ is holomorphic.

$\square$

Obvious from the implicit function theorem in the real analysis?

... No. One has to prove that $\varphi(z)$ is holomorphic.

<u>Proof</u>:

$f(w) := F(z_0, w)$: $f(w_0) = 0$, $f'(w_0) \neq 0$ by assumption.

$\Longrightarrow f$ has only one zero in a neighbourhood of w_0:

$$(\text{number of zeros in } |w - w_0| < \rho) = \frac{1}{2\pi i} \oint_{|w-w_0|=\rho} \frac{f'(w)}{f(w)}\, dw = 1$$

for sufficiently small ρ.

In general, if $|z - z_0|$ is so small that $F(z, w) \neq 0$ on $\{w \mid |w - w_0| = \rho\}$,

$$N(z) := \sharp\{w \mid F(z, w) = 0,\ |w - w_0| < \rho\} \qquad (\Rightarrow N(z) \in \mathbb{Z})$$

$$= \frac{1}{2\pi i} \oint_{|w-w_0|=\rho} \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)}\, dw. \qquad (\Rightarrow N(z) \text{ is continuous in } z.)$$

$\Longrightarrow N(z)$: locally constant.

We know $N(z_0) = 1$. $\Longrightarrow N(z) = 1$ if $|z - z_0| < r$ (r: small).

This means that the projection

$$\left\{ (z,w) \,\middle|\, \begin{array}{c} |z-z_0|<r, |w-w_0|<\rho \\ F(z,w)=0 \end{array} \right\} \ni (z,w) \mapsto z \in \{z \mid |z-z_0|<r\}$$

is bijective.

$$z \mapsto (z, \varphi(z)) : \text{ the inverse map, i.e., } F(z,\varphi(z)) = 0.$$

Formula in Complex Analysis:

- $g(w)$, $\psi(w)$: holomorphic on a neighbourhood of $\{w \mid |w-w_0| \leqq \rho\}$,
- $g(w) \neq 0$: on $\{w \mid |w-w_0| = \rho\}$,

Then

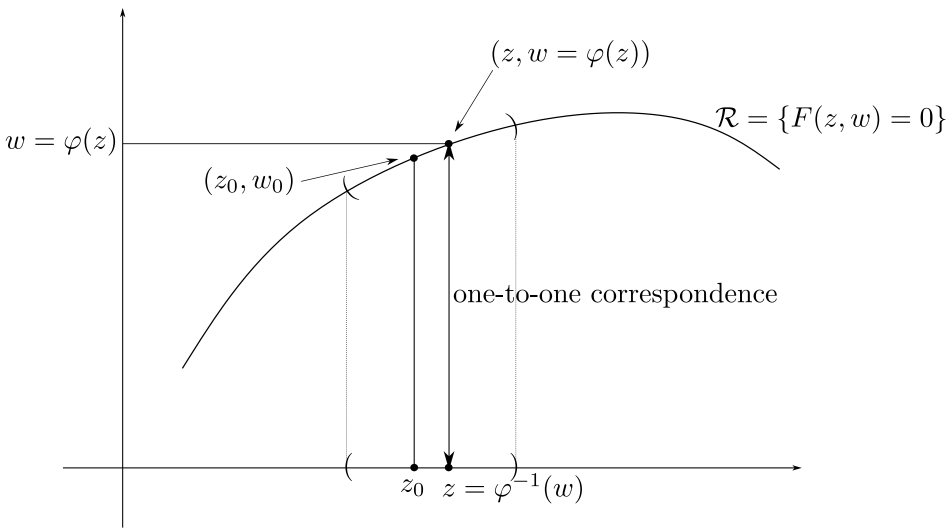
$$\sum_{\substack{w_i : g(w_i)=0 \\ |w_i - w_0|<\rho}} \psi(w_i) = \frac{1}{2\pi i} \oint_{|w-w_0|=\rho} \frac{g'(w)}{g(w)} \psi(w)\, dw.$$

Apply this formula to $g(w) = F(z, w)$ and $\psi(w) = w$:

$$\varphi(z) = \frac{1}{2\pi i} \oint_{|w-w_0|=\rho} \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)} w \, dw.$$

Integrand depends on z holomorphically. $\Longrightarrow \varphi(z)$: holomorphic. $\square$

$\dfrac{\partial F}{\partial w}(z_0, w_0) \neq 0 \Longrightarrow z$: a coordinate of $\mathcal{R} = \{F(z, w) = 0\}$ near (z_0, w_0):



$\dfrac{\partial F}{\partial z}(z_0, w_0) \neq 0 \Longrightarrow w$: a coordinate of $\mathcal{R} = \{F(z, w) = 0\}$ near (z_0, w_0).

$\dfrac{\partial F}{\partial w}(z_0, w_0) \neq 0$ and $\dfrac{\partial F}{\partial z}(z_0, w_0) \neq 0 \Longrightarrow z$ & w can be a coordinate.

Coordinate changes: $z \mapsto w = \varphi(z)$, $w \mapsto z = \varphi^{-1}(w)$ are holomorphic.

(Recall: the inverse of a holomorphic function is holomorphic.)

Summarising,

$\mathcal{R} = \{(z, w) \mid F(z, w) = 0\}$: one-dimensional complex manifold. $\square$

In algebraic geometry, it is called a *non-singular algebraic curve*:

- "non-singular": no singular points, where $\dfrac{\partial F}{\partial w} = \dfrac{\partial F}{\partial z} = 0$.
- "algebraic": F is a polynomial.
- "curve": one-dimensional over $\mathbb{C}$.

Example: $F(z,w) = w^2 - z$, $\mathcal{R} = \{(z,w) \mid w^2 = z\}$.

$$\frac{\partial F}{\partial w} = 2w, \qquad \frac{\partial F}{\partial z} = -1.$$

Hence,

- z: coordinate except at $(z,w) = (0,0)$.
- w: coordinate everywhere.

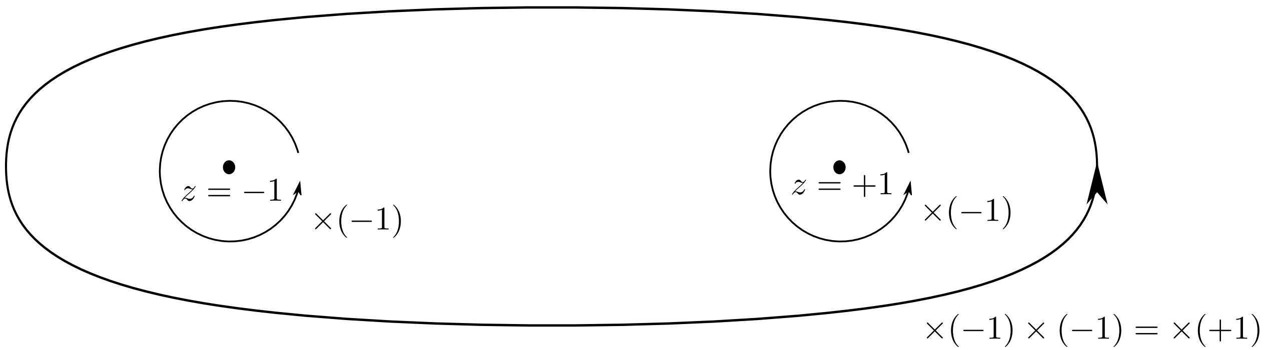
The function $\sqrt{z}$ on $\mathcal{R}$: $(z,w) \mapsto w$.

Defined everwhere! and holomorphic *even at* $z = 0$*!*

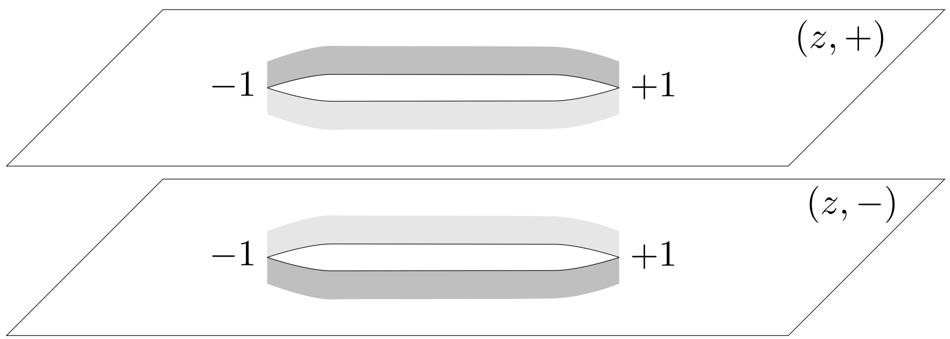
Riemann surface of $\sqrt{1-z^2}$.

$$f(z) := \sqrt{1-z^2} = \sqrt{(1-z)(1+z)}$$

- changes its sign when z goes around $+1$ or -1.
- does not change its sign when z goes aound both $+1$ and -1.



$\Longrightarrow$ Riemann surface of $f(z)$ = two $\mathbb{C}$'s cut along $[-1,+1]$ glued together.



$$\mathcal{R} = (\mathbb{C} \setminus \{\pm 1\})_+ \cup \{-1, +1\} \cup (\mathbb{C} \setminus \{\pm 1\})_-.$$

Another definition: $f(z)$ satisfies $f(z)^2 + z^2 - 1 = 0$. So,

$$\mathcal{R} = \{(z, w) \mid F(z, w) := z^2 + w^2 - 1 = 0\}.$$

Since

$$\frac{\partial F}{\partial w} = 2w, \qquad \frac{\partial F}{\partial z} = 2z,$$

- z is a coordinate around (z_0, w_0), $w_0 \neq 0$, i.e., $z_0 \neq \pm 1$.
- w should be used as a coordinate around $(\pm 1, 0)$.

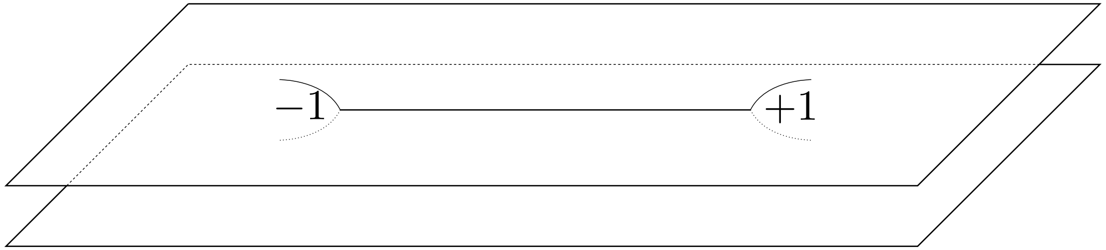
The function $f(z) = \sqrt{1 - z^2}$ is defined as

$$f : \mathcal{R} \ni (z, w) \mapsto w$$

on $\mathcal{R}$ as a *single-valued* function.

What surface is $\mathcal{R}$ topologically?

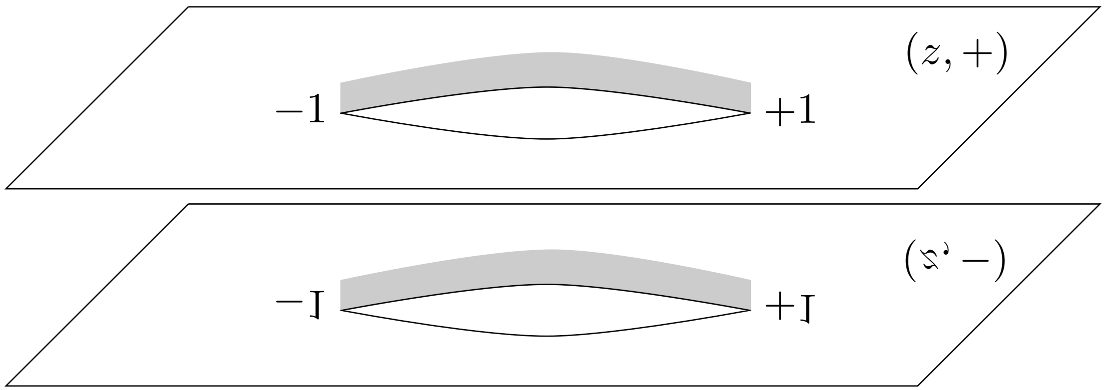
In the picture of $\mathcal{R}$ as glued $\mathbb{C}$'s:



the interval $[-1, +1]$ seems to be a self-intersection. But it is *NOT*!

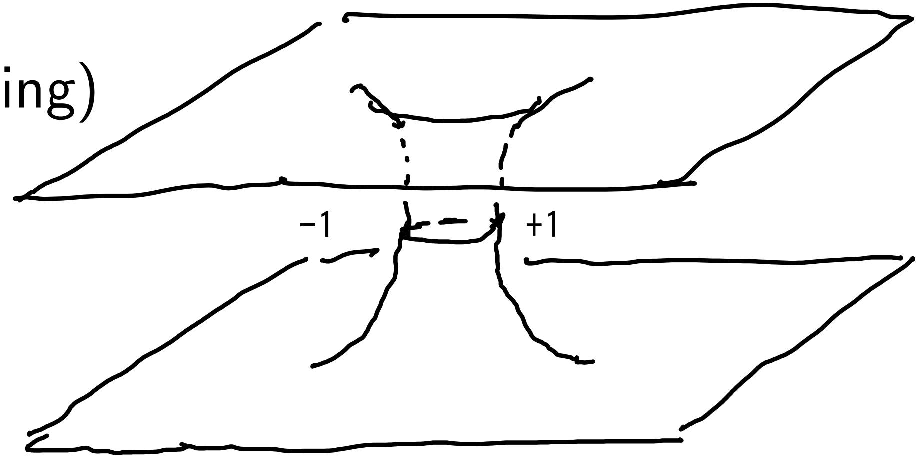
$\exists$ TWO points $(z, w) = (z, \pm\sqrt{1 - z^2})$ for each $z \in [-1, +1]$.

$\Longrightarrow$ Better to glue them with different orientations.

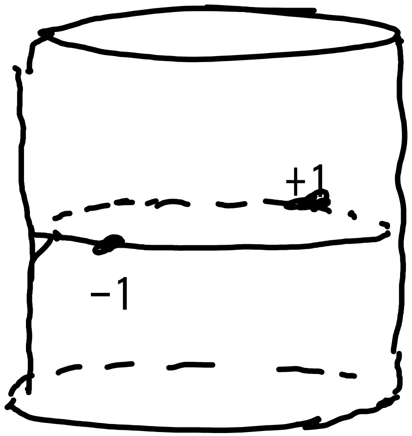


(Figure of gluing)

$\sim$



$\sim$



$=$ cylinder!

Recall: we want to study elliptic integrals with complex variables.

Prototype: $\displaystyle\int \frac{dz}{\sqrt{1-z^2}}$.

Question: Where does the 1-form $\omega = \dfrac{dz}{\sqrt{1-z^2}}$ live?

Answer: on the Riemann surface $\mathcal{R}$ of $\sqrt{1-z^2}$.

There we have to replace $\sqrt{1-z^2}$ by w: $\omega = \dfrac{dz}{w}$.

$\Longrightarrow \omega$ is not defined when $w=0$, i.e., $z=\pm 1$. ..., *NO!*

Recall that at $(\pm 1, 0) \in \mathcal{R}$ we have to use w as a coordinate.

$$w^2 = 1 - z^2 \xrightarrow{\frac{d}{dz}} 2w dw = -2z dz.$$

$$\Longrightarrow \ \omega = \frac{1}{w} dz = \frac{1}{w} \frac{-w\, dw}{z} = \frac{dw}{z} = \frac{-dw}{\sqrt{1-w^2}}: \text{ holomorphic at } (\pm 1, 0).$$

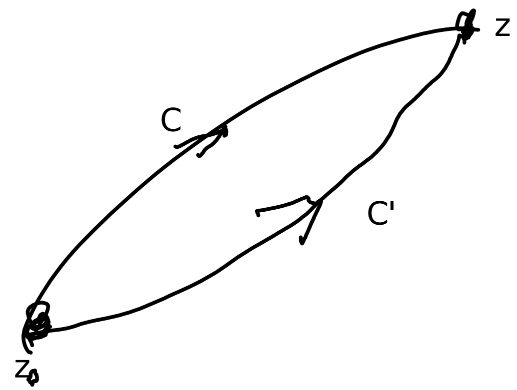
$$\omega = \frac{dz}{\sqrt{1-z^2}} = \frac{dz}{w} = \frac{-dw}{z}$$: holomorphic 1-form on the whole $\mathcal{R}$.

Recall: If $f(z)$ is an entire function ($=$ holomorphic on the whole $\mathbb{C}$), the indefinite integral

$$F(z) := \int_{z_0}^{z} f(z')\,dz'$$

defines a single-valued holomorphic function by virtue of Cauchy's integral theorem: (Figure $z_0 \xrightarrow{C \to C'} z$)

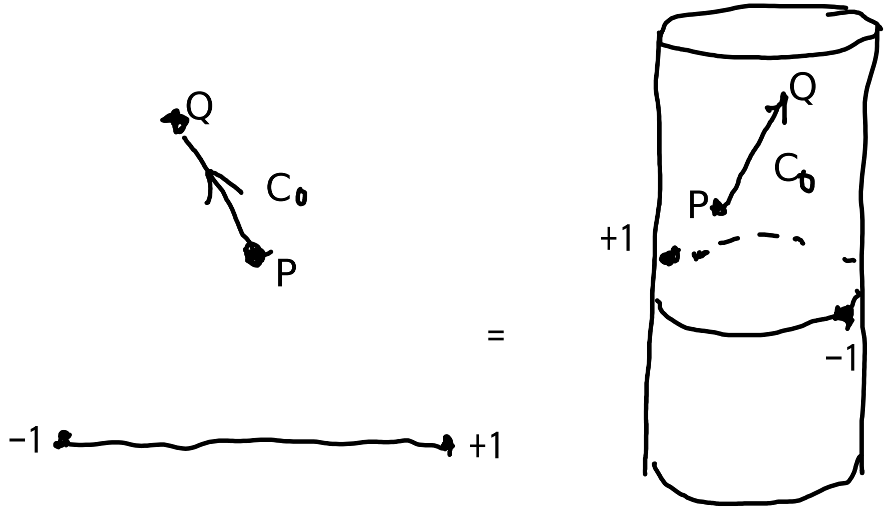
$$\int_C f(z)\,dz = \int_{C'} f(z')\,dz'.$$



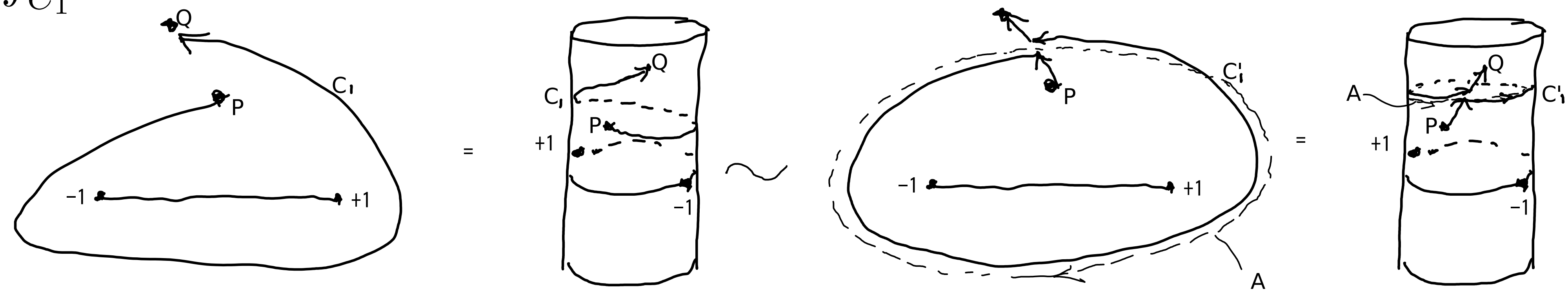
How about the integral of $\omega = \dfrac{dz}{\sqrt{1-z^2}}$?

Because of the non-trivial topology of $\mathcal{R}$, $\int_C \omega$ <u>depends</u> on C.

$\int_{C_0} \omega$: (Figure of C_0)

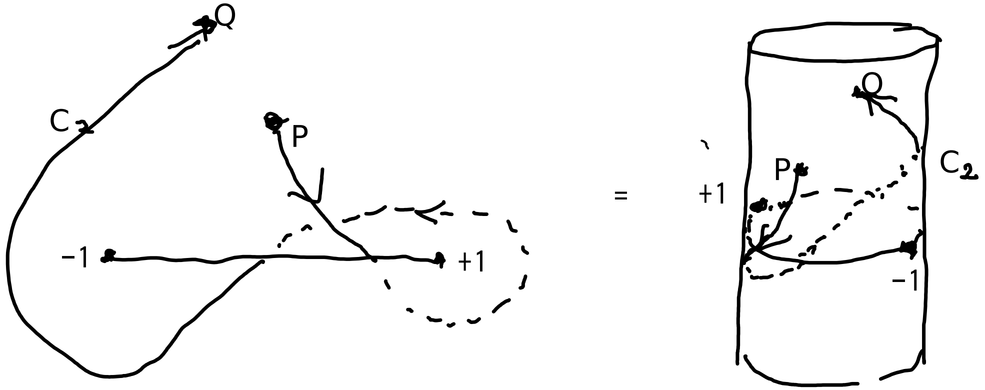


$\int_{C_1} \omega$: (Figure of C_1)

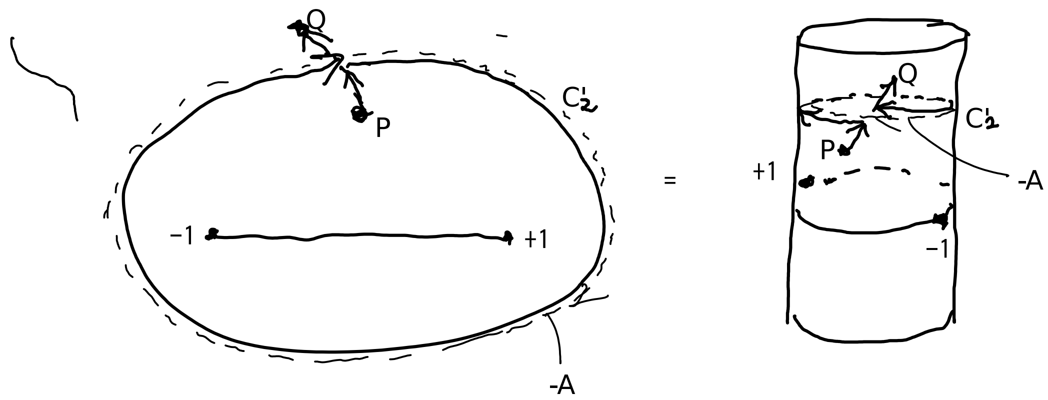


$$\Longrightarrow \int_{C_1} \omega - \int_{C_0} \omega = \int_A \omega.$$

$\int_{C_2} \omega$: (Figure of C_2)



$$\Longrightarrow \int_{C_2} \omega - \int_{C_0} \omega = -\int_A \omega.$$



For general contours? — Better to use terminology in topology.

The first homology group of a topological space X: (*very rough* summary)

$$H_1(X,\mathbb{Z}) := \langle \text{Free abelian group generated by closed curves in } X \rangle \,/\, \sim .$$

The equivalence relation: for closed curves C, C',

$$[C] \sim [C'] \iff C^{-1}C' = \bigcup (\text{boundaries of domains}).$$

("C and C' are homologically equivalent").

Figure: homological equivalence.

- homotopically equivalent $\Longrightarrow$ homologically equivalent.
- $H_1(X,\mathbb{Z})$: an abelian group.

Using this terminology:

$$\mathcal{R} \sim \text{ cylinder } \implies H_1(\mathcal{R}, \mathbb{Z}) = \mathbb{Z}[A].$$

Previous examples:

$$[C_1] - [C_0] = [A] \text{ in } H_1(\mathcal{R}, \mathbb{Z}) \qquad \implies \int_{C_1} \omega - \int_{C_0} \omega = \int_A \omega.$$

$$[C_2] - [C_0] = -[A] \text{ in } H_1(\mathcal{R}, \mathbb{Z}) \quad \implies \int_{C_1} \omega - \int_{C_0} \omega = -\int_A \omega.$$

In general,

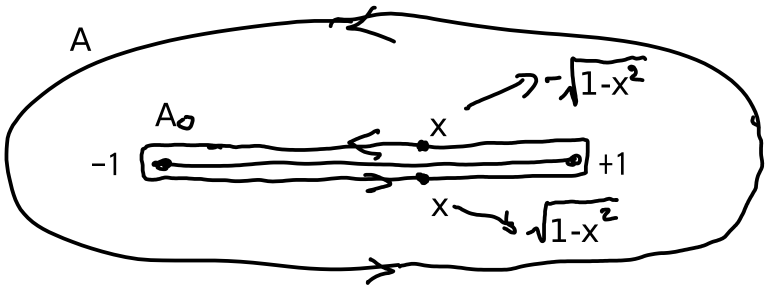
$$[C(P \to Q)] - [C_0] \in H_1(\mathcal{R}, \mathbb{Z}) = \mathbb{Z}[A]$$

$$\implies \int_{C(P \to Q)} \omega - \int_{C_0} \omega = n \int_A \omega, \quad n \in \mathbb{Z}$$

$\int_A \omega$: *period of* 1*-form* ω *over* A.

Shrink A to A_0: $\displaystyle\int_A \omega = \int_{A_0} \omega$.

(Figure of A_0: sign of $\sqrt{1-x^2}$ are different on each half plane.)



$$\begin{aligned}\int_{A_0} \omega &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + \int_{1}^{-1} \frac{-dx}{\sqrt{1-x^2}} \\ &= \arcsin x\big|_{x=-1}^{x=1} - \arcsin x\big|_{x=1}^{x=-1} \\ &= \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) - \left(\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right) = 2\pi.\end{aligned}$$

When P moves from $x \in \mathbb{C}$ and comes back to x,

$$u(P) = \int_0^P \omega$$

changes by $2\pi\times$(integer): $u(x) \rightsquigarrow u(x) + 2\pi n$, $n \in \mathbb{Z}$.

$\Longleftrightarrow$ the inverse function $x(u)$ of $u(x)$ has period 2π:

$$x(u + 2\pi n) = x(u), \qquad n \in \mathbb{Z}.$$

In fact,

$$u(x) = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \arcsin x, \qquad x(u) = \sin u.$$

"$\sin u$ is periodic because of the topology of the cylinder!"