Skoltech
Skolkovo Institute of Science and Technology

Course Syllabus

Course Title (in English)	Differential geometry of connections
Course Title (in Russian)	Дифференциальная геометрия связностей
Lead Instructor(s)	Kazarian, Maxim
Contact Person	Maxim Kazarian
Contact Person's E-mail	M.Kazarian@skoltech.ru

Course Description

In this course we present the basic concepts of modern differential geometry: metric, curvature, connection, etc. The goal of our study is to develop tools for practical efficient computations (including the art of manipulation with indices) supported by a deeper understanding of the geometric meaning of all notions and theorems. We will develop an approach based on the notion of connection with all its different aspects: covariant derivative, parallel transport, collection of Christoffel symbols, matrix-valued one-form etc.

Course Prerequisites / Recommendations	Students should be familiar with basic linear algebra and calculus including differential forms and Stokes theorem

Аннотация

В этом курсе мы ознакомим слушателей с основными концепциями современной дифференциальной геометрии: метрика, кривизна, связности, и т.д. Наша цель будет состоять в том, чтобы разработать инструменты для практических эффективных вычислений (включая технику манипуляций с индексами), сопровождая эти вычисления глубоким геометрическим содержанием всех используемых понятий и теорем. Наш подход будет основан на понятии связности в различных проявлениях: ковариантная производная, параллельный перенос, совокупность символов Кристоффеля, матричнозначная один-форма и т.д.

Course Academic Level	Master-level course suitable for PhD students
Number of ECTS credits	6

Topic	Summary of Topic	Lectures (# of hours)	Seminars (# of hours)	Labs (# of hours)
Vector fields and differential forms	Manifolds, charts, tangent and cotandent spaces. Vector fields and diffrential forms, coordinate presentation and its transformation under changes of coordinates	1	4	
Differential geometry of surfaces	The first and the second quadratic forms, principal curvatures, Gaussian curvature	1	5	
Curvature of a plane metric	Curvature of a plane metric. Theorema egregium.	2	6	
Gauss-Bonnet formula.	Connection and curvature forms. Parallel translate of vectors on a surface. Local Gauss-Bonnet formula. Global formula.	2	6	
Connection	Connection as a parralel translate nad connection as covariant derivative. Connection matrix. Guage group transformation. Curvature tensor. Cartan structure equation	2	6	
Riemannian manifolds	Levi-Cevita connection. Riemann curvature tensor. Ricci tensor and the scalar curvature. Symmetries of the Riemann tensor	2	6	
Geodesics	Variational interpretation. Exponential map. Normal coordinates. Conjugate points. Global geometry of Riemannian manifolds	1	5	

Assignment Type	Assignment Summary
Problem Set	Problem sets will include problems from all topics of the course. Some problems will be discussed in the classes but most of them students should solve by themselves and present written solutions.
Midterm Exam	Written exam. Students will be given a number of problems couvering first half of the course that they should be ready to solve in a given time (about 3 hours)
Final Exam	Written exam. Students will be given a number of problems covering the whole course that they should be ready to solve in a given time (about 3 hours)

Type of Assessment Graded

Grade Structure

Activity Type	Activity weight, %
Problem Set	30
Midterm Exam	30
Final Exam	40

A: 86

B: 76

C: 66

D:	56
E:	46
F:	0
Attendance Requirements	Mandatory with Exceptions

Maximum Number of Students

	Maximum Number of Students
Overall:	10
Per Group (for seminars and labs):	

Course Term (in context of Academic Year)	Term 1 Term 2
Course Delivery Frequency	Every two years

Students of Which Programs do You Recommend to Consider this Course as an Elective?

Masters Programs	PhD Programs
Mathematical and Theoretical Physics	Mathematics and Mechanics

Course Tags	Math Physics

Required Textbooks	ISBN-13 (or ISBN-10)
Dubrovin, B.A., Fomenko, A.T., Novikov, S.P., Modern Geometry — Methods and Applications, Springer, 1992	978-1-4684-9946-9

Recommended Textbooks	ISBN-13 (or ISBN-10)
Differential geometry : connections, curvature, and characteristic classes, Tu, L. W., ISBN: 9783319550824, 2017	9783319550824

Knowledge
Learning different aspects of the notion of connection and interaction between them. Lerning formalism of manipulating with indices and passing from martix to multiindex notationsand structures
Learning methods for finding geodesics and their usage in the local and global study of Rienmann manifolds
Lerning important examples of Riemann manifolds: sphere, Lobachevski plane in Klein and Poincare models

Skill
Ability of manipulating with vector and covector fields in arbitrary local coordinates; computation of commutator of fields and exteriour differential of forms
Ability to compute the curvature of a given metric on the plane.

Experience
Computation of areas and lengths, finding of principal and Gaussian curvatures of surfaces in the Euclidean three-space
Understanding of relationship between the local and global invarians of surfaces

Select Assignment 1 Type	Problem Set
Input Example(s) of Assignment 1 (preferable)	(1) Compute the curvature and the torsion of the spiral curve $(cos(a t),sin(a t), bt)$. (2) Find the focal set of (a) parabola $y=x^2$, (b) ellipsis $frac{x^2}{a^2}+frac{y^2}{b^2}=1$. (3) Compute the first, the second quadratic forms, the principal and the Gaussian curvature of (a) the sphere of radius R; (b) the paraboloid $y=x^2+y^2$. (4) Compute the curvature of the following metrics (a) $dx^2+x^2 dy^2$; (b) Conformal metric $h(x,y) (dx^2+dy^2)$ where h is a positive function; (c) $dx^2+2cos(omega) dx dy+dy^2$ where $omega(x,y) is a smooth function. (5) Compute the parallel translate along the parallel on the sphere of altitude phi in three different ways: (a) solving the equation of parallel translate (b) using the envelope (c) applying the Gauss-Bonnet formula (6) Compute the Christoffel symbols of the Levi-Cevita connection, the components of the Riemann curvature tensor, the Ricci tensor and the scalar curvature of the metrics from the problems (3) and (4)
Assessment Criteria for Assignment 1	Evaluated in points proportianal to the number of correctly solved problems
Select Assignment 2 Type	Midterm Exam
Input Example(s) of Assignment 2 (preferable)	The torus T is obtained by rotating a circle of radius r around a line lying in the plane of the circle on the distance R from its center. (1) Find a parametrization of this torus. (2) Compute the first and the second quadratic forms. (3) Find principal curvatures and the Gaussian curvature (4) Compute the curvature of the metric and compare the result with the Gaussian curvature computed earlier. (5) Compute the area of the torus and the integral of the Gaussian curvature over it. (6) Compute the parallel transport transformation along the parallel on the torus with the altitude phi.

Assessment Criteria for Assignment 2	Evaluated in points proportianal to the number of correctly solved problems
Select Assignment 3 Type	Final Exam
Input Example(s) of Assignment 3 (preferable)	(1) A closed non-selfintersecting curve has length ell. Compute the area of the strip of points situated on the distance at most $epsilon$ from the curve, where $epsilon$ is small (2) Prove that the length of the circle of radius r and its area on riemannian surface are given by $ell=2pi r+A r^3+O(r^4)$, $s=pi r^2+B r^4+O(r^5)$ for some constants A and B. Express A and B in terms of the curvature of the metric at the center of the circle. (3) The envelope of a space curve is the union of its tangent lines. Prove that the metric on any envelope is flat. Find Euclidean coordinates on the envelope of the spiral curve $(cos t, sin t, t)$. (4) Compute and draw the focal set of an ellipsoid.
Assessment Criteria for Assignment 3	Evaluated in points proportianal to the number of correctly solved problems