Elliptic Functions

Elliptic curves

§5.1 Riemann surfaces of $\sqrt{\varphi(x)}$, $\deg \varphi = 3, 4$

Want: elliptic integrals $\displaystyle\int R(x, \sqrt{\varphi(x)})\, dx$ with complex variables.

$\Longrightarrow$ Need: the Riemann surface $\mathcal{R}$ of $\sqrt{\varphi(x)}$, $\deg \varphi = 3, 4$.

The construction is the same as the case of $\sqrt{z}$, $\sqrt{1 - z^2}$.

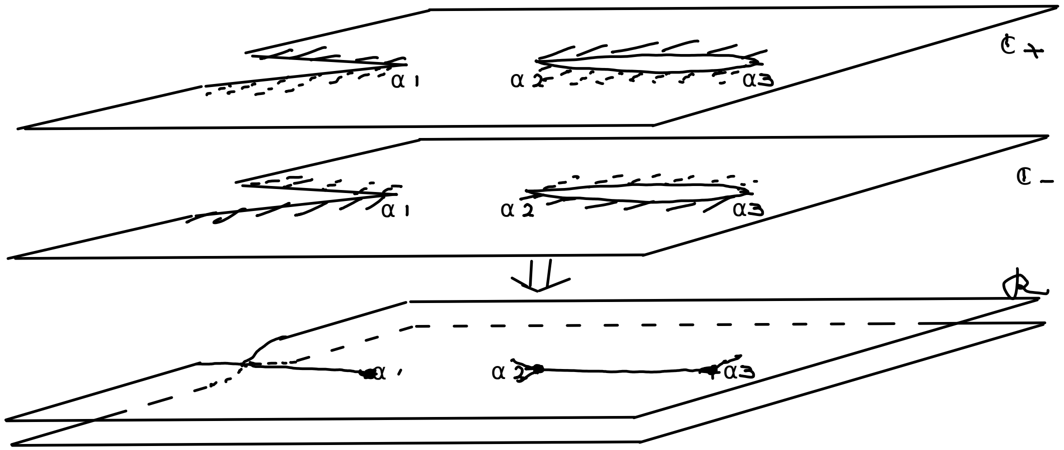
• $\deg \varphi(x) = 3$.

$$\varphi(z) = a(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)$$

$\alpha_1, \alpha_2, \alpha_3$: distinct, $a \neq 0$.

The Riemann surface $\mathcal{R}$ of $\sqrt{\varphi(x)}$

$$= \text{two copies of } \mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\} \text{ glued } \cup \{\alpha_1, \alpha_2, \alpha_3\}.$$



$$\mathcal{R} = (\mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\})_+ \cup \{\alpha_1, \alpha_2, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\})_-$$

$$= \{(z, w) \mid w^2 = \varphi(z)\}.$$

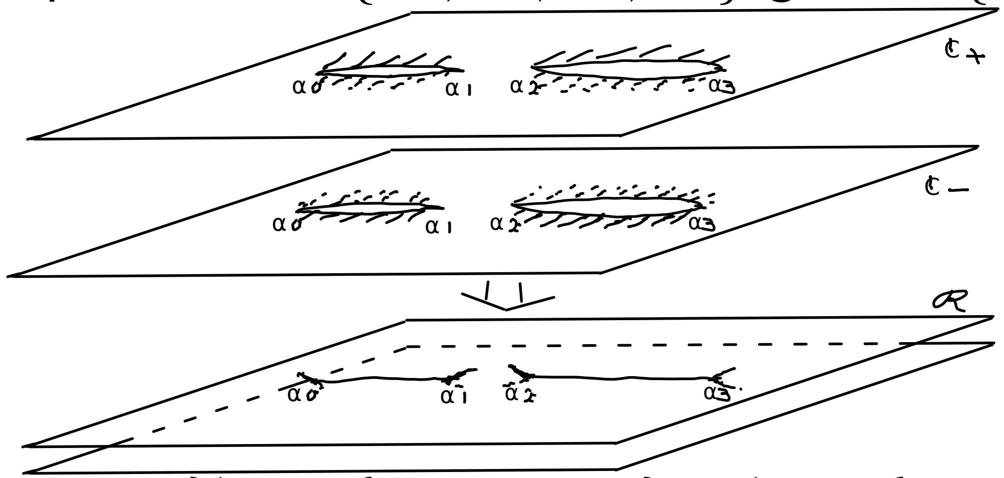
• $\deg \varphi(x) = 4$.

$$\varphi(z) = a(z-\alpha_0)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)$$

$a \neq 0$, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$: distinct.

The Riemann surface $\mathcal{R}$ of $\sqrt{\varphi(x)}$

$=$ two copies of $\mathbb{C} \setminus \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ glued $\cup\{\alpha_1, \alpha_2, \alpha_3\}$.



$$\mathcal{R} = (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_+ \cup \{\alpha_0, \dots, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_-$$

$$= \{(z, w) \mid w^2 = \varphi(z)\}.$$

Proposition:

For both cases, $\deg \varphi(z) = 3$ and 4,

(i) $\mathcal{R} = \{(z,w) \mid w^2 = \varphi(z)\}$: a non-singular algebraic curve.

($\Longleftrightarrow \left(F, \dfrac{\partial F}{\partial z}, \dfrac{\partial F}{\partial w}\right) \neq (0,0,0)$, where $F(z,w) = w^2 - \varphi(z)$).

(ii) $\sqrt{\varphi(z)} = w$: holomorphic on $\mathcal{R}$.

(iii) 1-form $\omega = \dfrac{dz}{\sqrt{\varphi(z)}} = \dfrac{dz}{w}$: holomorphic on $\mathcal{R}$.

Exercise: Check these statements.

§5.2 Compactification and elliptic curves

When $\deg \varphi(z) > 2$, $\displaystyle\int_{z_0}^{\infty} \frac{dz}{\sqrt{\varphi(z)}}$ converges.

$\Longrightarrow$ Need to add ∞ to $\mathcal{R}$ (*Compactification*).

• $\deg \varphi = 3$.

Use the embedding into the *projective plane* $\mathbb{P}^2$:

$$\mathcal{R} = \{(z,w) \mid w^2 = \varphi(z)\} \subset \quad \mathbb{C}^2 \quad \hookrightarrow \qquad \mathbb{P}^2$$
$$(z,w) \mapsto [1 : z : w].$$

Recall:

$$\mathbb{P}^2 = \mathbb{C}^3 \setminus \{0\}/\sim,$$
$$(a,b,c) \sim (a',b',c') \Longleftrightarrow \exists \lambda \neq 0,\ (\lambda a, \lambda b, \lambda c) = (a',b',c').$$

Embedding of $\mathbb{C}^2$ into $\mathbb{P}^2$:

$$\mathbb{C}^2 \ni (z, w) \mapsto [1 : z : w] \in \mathbb{P}^2,$$

$$\mathbb{P}^2 \supset U_0 := \{[x_0 : x_1 : x_2] \mid x_0 \neq 0\} \ni [x_0 : x_1 : x_2] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \mathbb{C}^2.$$

$\mathcal{R}$ is a subset of $\mathbb{C}^2 \cong U_0 \subset \mathbb{P}^2$: $\left(\dfrac{x_2}{x_0}\right)^2 = \varphi\left(\dfrac{x_1}{x_0}\right)$, i.e.,

$$(*) \qquad x_0 x_2^2 = a(x_1 - \alpha_1 x_0)(x_1 - \alpha_2 x_0)(x_1 - \alpha_3 x_0).$$

Extend $\mathcal{R}$ by this equation:

$$\bar{\mathcal{R}} := \{[x_0 : x_1 : x_2] \mid (*)\} \subset \mathbb{P}^2.$$

What points are added to $\mathcal{R}$?

Since $\mathbb{P}^2 \smallsetminus U_0 = \{[x_0 : x_1 : x_2] \mid x_0 = 0\}$,

$$\begin{aligned}\bar{\mathcal{R}} \smallsetminus \mathcal{R} &= \{[x_0 : x_1 : x_2] \mid x_0 = 0,\ (*)\} \\ &= \{[x_0 : x_1 : x_2] \mid x_0 = 0 = a x_1^3\} \\ &= \{[x_0 : x_1 : x_2] \mid x_0 = x_1 = 0\} = \{[0 : 0 : 1]\}\end{aligned}$$

Namely, $\quad \bar{\mathcal{R}} = \mathcal{R} \cup \{\infty\}$, $\infty = [0 : 0 : 1]$.

The coordinates of $\mathbb{P}^2$ in the neighbourhood of ∞: $(\xi, \eta) := \left(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}\right)$.

$$\begin{aligned}(*) \iff & \frac{x_0}{x_2} = a\left(\frac{x_1}{x_2} - \alpha_1 \frac{x_0}{x_2}\right)\left(\frac{x_1}{x_2} - \alpha_2 \frac{x_0}{x_2}\right)\left(\frac{x_1}{x_2} - \alpha_3 \frac{x_0}{x_2}\right) \\ \iff & \xi = a(\eta - \alpha_1 \xi)(\eta - \alpha_2 \xi)(\eta - \alpha_3 \xi).\end{aligned}$$

Exercise:

Check that the equation

$$\xi = a(\eta - \alpha_1\xi)(\eta - \alpha_2\xi)(\eta - \alpha_3\xi)$$

defines a non-singular algebraic curve in the nbd of $(\xi, \eta) = (0, 0)$.

$$\left.\begin{array}{l} \bar{\mathcal{R}} = \text{defined by equation } (*) \implies \text{closed in } \mathbb{P}^2 \\ \mathbb{P}^2\text{: compact} \end{array}\right\} \implies \bar{\mathcal{R}} \text{ : compact.}$$

$\implies \bar{\mathcal{R}}$ is a compact Riemann surface, a *compactification* of $\mathcal{R}$.

• <u>$\deg \varphi = 4.$</u>

$$\varphi(z) = a(z-\alpha_0)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3).$$

Want: compactification of $\mathcal{R} = \{(z,w) \mid w^2 - \varphi(z) = 0\}$.

Try the same procedure as before:

$$\begin{array}{cccc} \mathcal{R} \subset & \mathbb{C}^2 & \hookrightarrow & \mathbb{P}^2 \\ & & (z,w) \mapsto [1:z:w]. & \end{array}$$

The homogeneous equation for $\mathcal{R}$:

$$\left(\frac{x_2}{x_0}\right)^2 - a\left(\frac{x_1}{x_0}-\alpha_0\right)\left(\frac{x_1}{x_0}-\alpha_1\right)\left(\frac{x_1}{x_0}-\alpha_2\right)\left(\frac{x_1}{x_0}-\alpha_3\right) = 0,$$

$$\text{i.e., } x_0^2x_2^2 - a(x_1-\alpha_0x_0)\cdots(x_1-\alpha_3x_0) = 0. \qquad ((**))$$

As before $\{[x_0 : x_1 : x_2] \mid (**)\} = \mathcal{R} \cup \{\infty = [0:0:1]\}$.

Alas! $\infty = [0:0:1]$ is a singular point!

Exercise: Check this.

Another compactification:

Instead of $\mathbb{P}^2$, use $X := W \cup W'/\sim$, where

$$W = \mathbb{C}^2 \ni (z, w), \qquad W' = \mathbb{C}^2 \ni (\xi, \eta),$$

$$(z, w) \sim (\xi, \eta) \Longleftrightarrow z\xi = 1,\ w = \frac{\eta}{\xi^2}$$

$\mathcal{R} \subset W$ as before. $\Longrightarrow$ the equation of $\mathcal{R} \cap W'$:

$$\left(\frac{\eta}{\xi^2}\right)^2 - a\left(\frac{1}{\xi} - \alpha_0\right)\left(\frac{1}{\xi} - \alpha_1\right)\left(\frac{1}{\xi} - \alpha_2\right)\left(\frac{1}{\xi} - \alpha_3\right) = 0,$$

$$\text{i.e., } \eta^2 - a(1-\alpha_0\xi)(1-\alpha_1\xi)(1-\alpha_2\xi)(1-\alpha_3\xi) = 0.$$

$$\mathcal{R}' := \{(\xi, \eta) \in W' \mid \eta^2 = a(1-\alpha_0\xi)(1-\alpha_1\xi)(1-\alpha_2\xi)(1-\alpha_3\xi)\}.$$

a non-singular algebraic curve as before.

$$\bar{\mathcal{R}} := \mathcal{R} \cup \mathcal{R}' \subset X = W \cup W'.$$

<u>What is $\bar{\mathcal{R}}$?</u>

What point lies in $\bar{\mathcal{R}} \setminus \mathcal{R}$?

$$W' \setminus W = \{(\xi = 0, \eta) \mid \eta \in \mathbb{C}\}$$
$$\Longrightarrow \mathcal{R}' \setminus \mathcal{R} = \{(0, \eta) \mid \eta^2 = a\} = \{(0, \pm\sqrt{a})\} \subset W'.$$

They do not belong to W, i.e., they are "infinities": $\infty_\pm := (0, \pm\sqrt{a})_{W'}$.

$$\bar{\mathcal{R}} = \mathcal{R} \cup \{\infty_+, \infty_-\}.$$

Interpretation of $\bar{\mathcal{R}}$ by the gluing construction:

$$\mathcal{R} = (\text{the Riemann surface of } w = \sqrt{\varphi(z)})$$
$$= (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_+ \cup \{\alpha_0, \dots, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_-.$$

- When $\alpha_i \neq 0$ for $\forall i = 0, \dots, 3$.

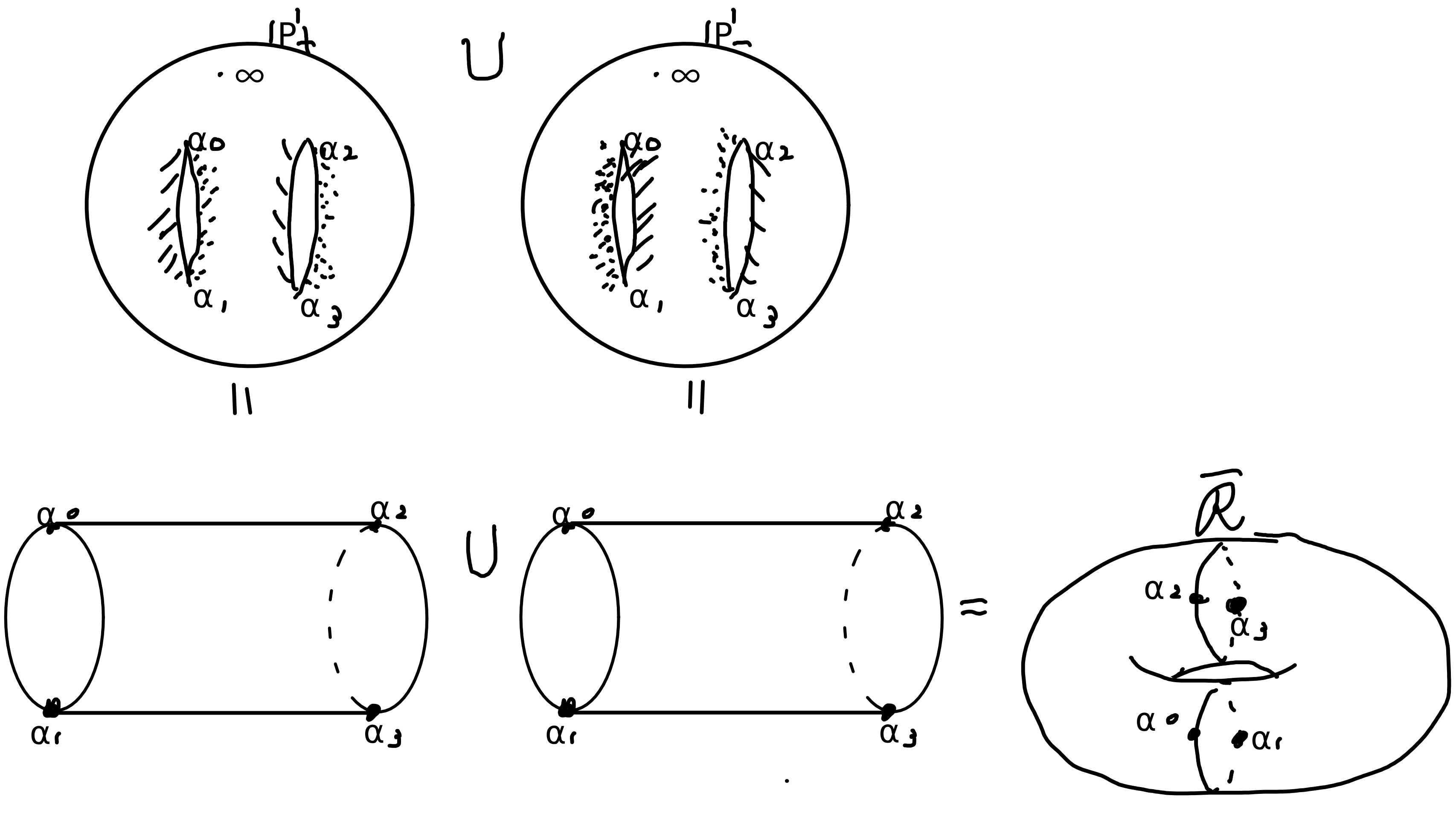
Denote $\beta_i := \alpha_i^{-1}$.

$$\mathcal{R}' = (\text{the Riemann surface of } \eta = \sqrt{a(1-\alpha_0\xi)\cdots(1-\alpha_3\xi)})$$
$$= (\mathbb{C} \setminus \{\beta_0, \dots, \beta_3\})_+ \cup \{\beta_0, \dots, \beta_3\} \cup (\mathbb{C} \setminus \{\beta_0, \dots, \beta_3\})_-.$$
$$\begin{cases} \xi_+ \longleftrightarrow z_+ = \frac{1}{\xi_+} \\ 0_+ \longleftrightarrow \infty_+ \end{cases}, \quad \beta_i \longleftrightarrow \alpha_i = \frac{1}{\beta_i}, \ \begin{cases} \xi_- \longleftrightarrow z_- = \frac{1}{\xi_-} \\ 0_- \longleftrightarrow \infty_- \end{cases},$$

$\Longrightarrow$ $\bar{\mathcal{R}}$ is constructed by gluing two $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$'s together.

$$\bar{\mathcal{R}} = (\mathbb{P}^1 \smallsetminus \{\alpha_0, \ldots, \alpha_3\})_+ \cup \{\alpha_0, \ldots, \alpha_3\} \cup (\mathbb{P}^1 \smallsetminus \{\alpha_0, \ldots, \alpha_3\})_-.$$

Figure: two $\mathbb{P}^1$'s with cuts $\alpha_0\alpha_1$ & $\alpha_2\alpha_3$ glued together $\cong$ a torus:



- When one of α_i's (say α_0) $= 0$.

$$\mathcal{R}' = (\text{the Riemann surface of } \eta = \sqrt{a(1-\alpha_1\xi)(1-\alpha_2\xi)(1-\alpha_3\xi)})$$
$$= (\mathbb{C} \setminus \{\beta_1, \beta_2, \beta_3\})_+ \cup \{\beta_1, \beta_2, \beta_3\} \cup (\mathbb{C} \setminus \{\beta_1, \beta_2, \beta_3\})_-.$$

$\Longrightarrow$ everything is the same as before.

Definition: *Elliptic curve*: compactification of $\{(z, w) \in \mathbb{C}^2 \mid w^2 = \varphi(z)\}$, $\deg \varphi(z) = 3$ or 4.

Remark: When $\deg \varphi \geq 5$: *hyperelliptic curve*.

Recall: Elliptic integrals are reduced to

$$\int R(x, \sqrt{(1-x^2)(1-k^2x^2)})\, dx$$

by means of fractional linear transformations.

The same is true for elliptic curves:

Any elliptic curves are isomorphic to

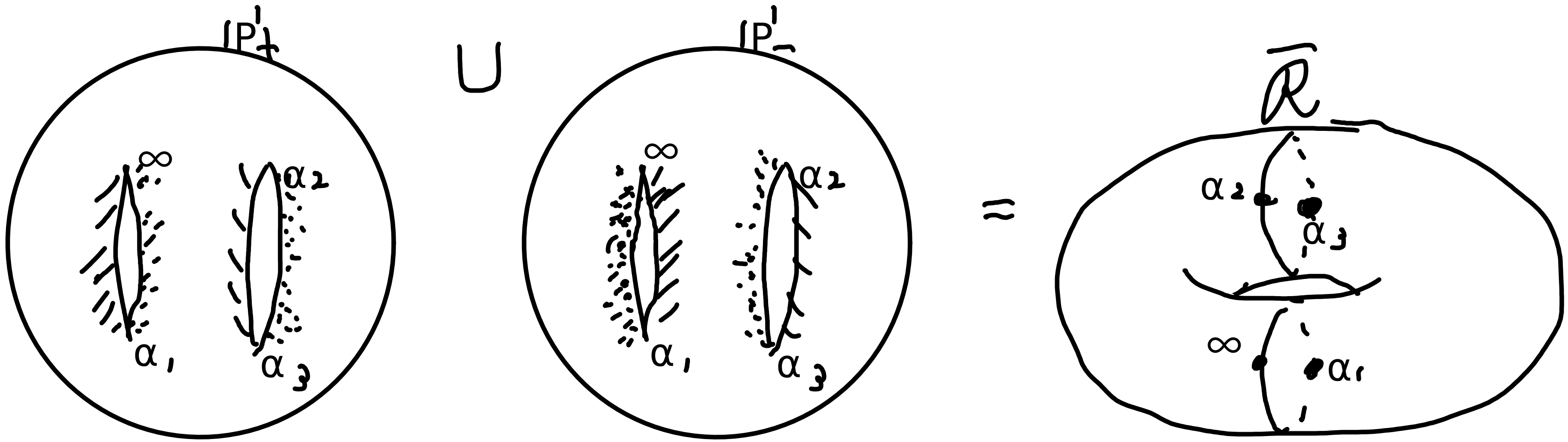
$$\overline{\{(z,w) \mid w^2 = (1-z^2)(1-k^2z^2)\}}^{\text{compactification}}, \qquad k \in \mathbb{C}$$

as Riemann surfaces $=$ 1-dim. complex manifold.

Exercise*: Prove this.

In particular, any elliptic curve is homeomorphic to a torus.

Gluing of $\mathbb{P}^1$'s when $\deg \varphi = 3$: $\varphi(z) = a(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$.



Cuts are $\infty\alpha_1$ and $\alpha_2\alpha_3$.