Elliptic Integrals and Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:
$$\text{(your final mark)} = \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points}}, 10\right\}$$
- About fifteen problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **9** – **11**: 2 November 2021.

9. (1 pt.) Let $\varphi(z)$ be a polynomial satisfying the conditions in the seminar (26 October 2021). Show the following:

(i) $\mathcal{R}_\varphi := \{(z,w) \in \mathbb{C}^2 \mid F_\varphi(z,w) := w^2 - \varphi(z) = 0\}$ is a non-singular algebraic curve over $\mathbb{C}$.

(ii) The 1-form $\omega := \dfrac{dz}{w}$ is holomorphic everywhere on $\mathcal{R}_\varphi$.

10. Show that the closure $\bar{\mathcal{R}}_\varphi$ of $\mathcal{R}_\varphi$ in $\mathbb{P}^2(\mathbb{C})$ constructed in the seminar on 26 October 2021 is

(i) (1 pt.) *non-singular* if $\deg \varphi(z) = 3$.

(ii) (1 pt.) *singular* if $\deg \varphi(z) = 4$.

11. (2 pt.) Show that the elliptic curve $\bar{\mathcal{R}}_\varphi$ ($\deg \varphi = 3$ or 4) is isomorphic to $\bar{\mathcal{R}}_\psi$, $\psi(z) = (1-z^2)(1-k^2z^2)$ for some $k \in \mathbb{C} \setminus \{0, \pm 1\}$. Construct a biholomorphic bijection $\Phi : \bar{\mathcal{R}}_\varphi \xrightarrow{\sim} \bar{\mathcal{R}}_\psi$ *explicitly.* (Hint: use the result of **5** (ii).)