Elliptic Integrals and Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$(\text{your final mark}) = \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points}}, 10\right\}$$

- About fifteen problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **12** – **13**: 16 November 2021.

12. (1 pt.) Check that the Abelian differential ω_1 defined in the seminar (11 November 2021) is holomorphic and nowhere vanishing on the ellipitc curve $\bar{\mathcal{R}}_\varphi$ for a polynomial $\varphi(z)$ of degree 3.

13. Let the A-cycle and the B-cycle on the elliptic curve $\bar{\mathcal{R}}_\varphi$ ($\varphi(z) = (1-z^2)(1-k^2z^2)$) be those defined in the seminar on 11 November 2021 and $\omega_1 := \dfrac{dz}{w}$, $\omega_2 := \sqrt{\dfrac{1-k^2z^2}{1-z^2}}dz$. We assume $0 < k < 1$.

(i) (1 pt.) Prove that the A-period of the Abelian differential ω_2 is equal to $4E(k)$, where $E(k)$ is the complete elliptic integral of the second kind.

(ii) (1 pt.) Show $d\left(\dfrac{zw}{1-k^2z^2}\right) = \dfrac{k^2z^4 - 2z^2 + 1}{1-k^2z^2}\omega_1$. (Hint: Use $w^2 = \varphi(z)$ to compute dw.)

(iii) (2 pt.) Prove that $K(k)$ (the complete elliptic integral of the first kind) and $E(k)$ satisfy the following system of differential equations:

$$\frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}, \qquad \frac{dK}{dk} = \frac{E}{kk'^2} - \frac{K}{k}.$$

(Hint: To obtain the first, differentiate ω_2 with respect to k and integrate it over $[0,1]$. To obtain the second, compare $\dfrac{\partial}{\partial k}\omega_1 - \dfrac{1}{kk'^2}\omega_2 + \dfrac{1}{k}\omega_1$ with (ii) and consider the A-period. Note that the integral of an exact form over a cycle is zero.)