Elliptic Functions

Elliptic functions (general theory)

§8.1 Definition of Elliptic Functions

At last we have come to the definition of elliptic functions!

Definition.

A meromorphic function on an elliptic curve is called an *elliptic function*.

A direct consequence:

f, g: elliptic functions $\Longrightarrow$ $f \pm g$, fg, f/g: elliptic functions.

Namely, $\{$elliptic functions$\}$ is a field.

By the Abel-Jacobi theorem

$$\text{Elliptic curve} \cong \mathbb{C}/\Gamma, \qquad \Gamma = \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B.$$

Ω_A, Ω_B: linearly independent over $\mathbb{R}$.

$\Longrightarrow$ An alternative (*standard*) definition:

A meromorophic function on $\mathbb{C}$ satisfying

$$f(u + \Omega_A) = f(u), \qquad f(u + \Omega_B) = f(u)$$

is called an *elliptic function* with periods Ω_A and Ω_B.

Remark: For any Ω_A, Ω_B, $\exists$ an elliptic curve, i.e.,

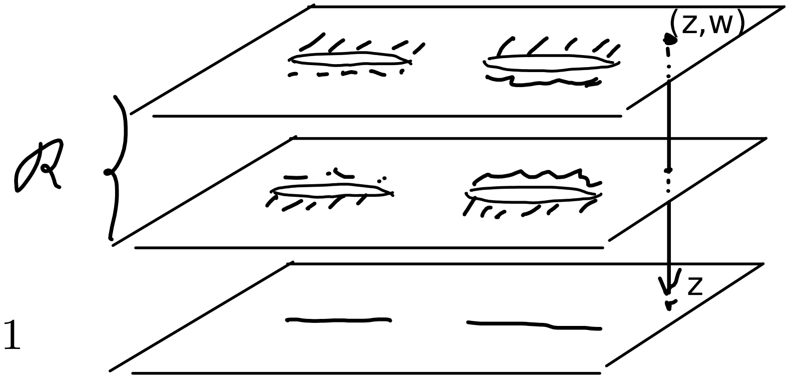
$\exists \varphi(z)$: polynomial of degree 3 or 4 such that

$$\mathbb{C}/\mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B \cong \overline{\{w^2 = \varphi(z)\}}.$$

Proved later. ($\Longleftarrow$ differential equation of $\wp(z)$)

Example:

$\varphi(z)$: as before.



$$\begin{array}{ccc} \bar{\mathcal{R}} = \overline{\{(z,w) \mid w^2 = \varphi(z)\}} & \xrightarrow{\mathrm{pr}} & \mathbb{P}^1 \\ (z,w) & \mapsto & z, \text{ (projection)} \\ \infty \text{ or } \infty_\pm & \mapsto & \infty. \end{array}$$

pr: a holomorphic map (a meromorphic function with a pole at ∞).

$\Longrightarrow$ Composition

$$\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma \xrightarrow{AJ^{-1}} \bar{\mathcal{R}} \xrightarrow{\mathrm{pr}} \mathbb{P}^1$$

gives an elliptic function on $\mathbb{C}$: $f(u) = \mathrm{pr} \circ AJ^{-1} \circ \pi(u)$.

Since AJ is defined by an elliptic integral, this means that

the inverse function of an elliptic integral is an elliptic function!

• $\underline{\varphi(z) = 4z^3 - g_2 z - g_3.\ (g_2, g_3 \in \mathbb{C})}$

Fix the base point of the Abel-Jacobi map to ∞:

$$AJ(z) = \int_\infty^z \frac{dz}{w} = \int_\infty^z \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

$\wp(u) := \mathrm{pr} \circ AJ^{-1} \circ \pi(u)$: *Weirstrass's* $\wp$ *function*.

$AJ(\infty) = 0 \Longrightarrow \wp(0) = \infty$, i.e., $u = 0$ is a pole.

• $\underline{\varphi(z) = (1 - z^2)(1 - k^2 z^2).\ (k \in \mathbb{C},\ k \neq 0, \pm 1)}$

Fix the base point of the Abel-Jacobi map to 0:

$$AJ(z) = \int_0^z \frac{dz}{w} = \int_0^z \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

$\mathrm{sn}(u) := \mathrm{pr} \circ AJ^{-1} \circ \pi(u)$: *Jacobi's* sn *function*.

A natural generalisation of the previously defined sn over $\mathbb{R}$.

• $\wp(u)$, $\mathrm{sn}(u)$: periodic with periods Ω_A, Ω_B. $\Longleftarrow \pi : \mathbb{C} \to \mathbb{C}/\Gamma$.

• For $\varphi(z) = (1 - z^2)(1 - k^2 z^2)$, we have computed

$$\Omega_A = 4K(k), \qquad \Omega_B = 2iK'(k).$$

$\Longrightarrow$ Periods of $\mathrm{sn}(u)$: $4K(k)$, $2iK'(k)$.

Consistent with the previous definition for $\mathrm{sn}(x)$, $x \in \mathbb{R}$.

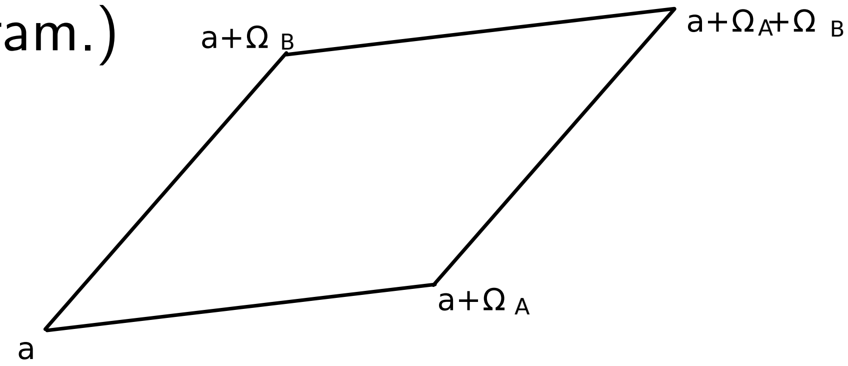
• We construct $\wp(u)$ and $\mathrm{sn}(u)$ by different methods later.

§8.2 General Properties of Elliptic Functions

$f(u)$: an elliptic function on $\mathbb{C}$ with periods Ω_A and Ω_B.

We call a parallelogram spanned by Ω_A & Ω_B a *period parallelogram*.

(Figure of a period parallelogram.)



Theorem (Liouville)

If an elliptic funtion $f(u)$ is entire, then $f(u)$ is constant.

Proof: f: doubly periodic. $\Longrightarrow f(\mathbb{C}) = f(\text{period parallelogram})$.

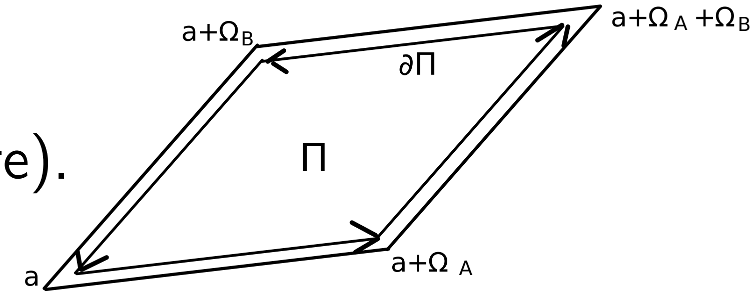
f: continuous & a period parallelogram is bounded. $\Longrightarrow f(\mathbb{C})$: bounded.

Liouville's theorem (Complex analysis!) $\Longrightarrow f$: constant. $\square$

Theorem (Liouville)

The sum of residues of $f(u)$ at poles in one period parallelogram is zero.

Proof:



Π: a period parallelogram (poles of $f \notin \partial\Pi$; cf. Figure).

$$\int_{\partial\Pi} f(u)\,du = 2\pi i(\text{the sum of residues in } \Pi).$$

On the other hand,

$$\int_{\partial\Pi} f(u)\,du = \left(\int_a^{a+\Omega_A} + \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} + \int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} + \int_{a+\Omega_B}^{a}\right) f(u)\,du.$$

By the periodicity,

$$\int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} f(u)\,du = \int_a^{a+\Omega_B} f(u)\,du = -\int_{a+\Omega_B}^{a} f(u)\,du,$$

$$\int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} f(u)\,du = \int_{a+\Omega_A}^{a} f(u)\,du = -\int_a^{a+\Omega_A} f(u)\,du.$$

Summing up, $2\pi i(\text{the sum of residues in }\Pi) = \int_{\partial\Pi} f(u)\,du = 0.$ $\square$

<u>Corollary</u>:

$\nexists$ an elliptic function with only one simple pole in a period parallelogram.

<u>Proof</u>:

Otherwise, the sum of residue $=$ the residue at the simple pole $\neq 0$. $\square$

<u>Remark</u>: We have already proved the same fact in the proof of the Abel-Jacobi theorem ($\nexists F(z)\omega_1$).

Definition:

order of $f = \operatorname{ord} f :=$ $\sharp$ poles with multiplicity in a period parallelogram.

Corollary means "There is no elliptic funtion of order 1."

For the next theorem, we need an obvious lemma:

Lemma: $f(u)$: an elliptic function $\Longrightarrow$ $f'(u)$: an elliptic function.

($f(u+\Omega_A) = f(u+\Omega_B) = f(u) \Longrightarrow f'(u+\Omega_A) = f'(u+\Omega_B) = f'(u)$)

Theorem:

For any $a \in \mathbb{C}$ and Π: a period parallelogram ($f \neq \infty, a$ on $\partial\Pi$).

$$\sharp \text{ of } \{u \in \Pi \mid f(u) = a\} \text{ with multiplicities} = \operatorname{ord} f.$$

Proof:

$f(u) - a$: an elliptic function of order $\operatorname{ord} f$. $\Longrightarrow$ May assume $a = 0$.

$$
\begin{aligned}
&\sharp\{\text{zeroes of } f(u) \text{ in } \Pi\} - \sharp\{\text{poles of } f(u) \text{ in } \Pi\} \\
&= \frac{1}{2\pi i} \oint_{\partial\Pi} \frac{f'(u)}{f(u)}\, du \qquad \text{(argument principle)} \\
&= 0. \qquad \begin{pmatrix} \text{lemma} \Rightarrow f'/f : \text{ elliptic function;} \\ \oint_{\partial\Pi} \text{(elliptcit function)}\, du = 0. \end{pmatrix}
\end{aligned}
$$

□

Theorem: $N := \operatorname{ord} f$, $a \in \mathbb{C}$.

$\alpha_1, \ldots, \alpha_N$: points in Π, $f(\alpha_i) = a$ (with multiplicities).

$\beta_1, \ldots, \beta_N$: poles of $f(u)$ in Π (with multiplicities).

$$\Longrightarrow \alpha_1 + \cdots + \alpha_N \equiv \beta_1 + \cdots + \beta_N \mod \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B.$$

Proof:

Again, we may assume $a = 0$.

Recall the generalised argument principle in complex analysis:

D: a domain, f: meromorphic, φ: holomorphic in a nbd of $\bar{D}$

$$\Longrightarrow \quad \frac{1}{2\pi i}\oint_{\partial D} \varphi(u)\frac{f'(u)}{f(u)}\,du = \sum_{\alpha\in D: f(\alpha)=0} \varphi(\alpha) - \sum_{\beta\in D:\ \text{pole of } f} \varphi(\beta).$$

Apply it to $D = \Pi$, $\varphi(u) = u$:

$$\frac{1}{2\pi i}\oint_{\partial\Pi} u\frac{f'(u)}{f(u)}\,du = \sum_{j=1}^{N}\alpha_j - \sum_{j=1}^{N}\beta_j.$$

NOTE: The integrand is *NOT* an elliptic function because of u!

Let us compute

$$\oint_{\partial\Pi} u\frac{f'(u)}{f(u)}\,du = \left(\int_a^{a+\Omega_A} + \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} + \int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} + \int_{a+\Omega_B}^{a}\right) u\frac{f'(u)}{f(u)}\,du.$$

The second term is

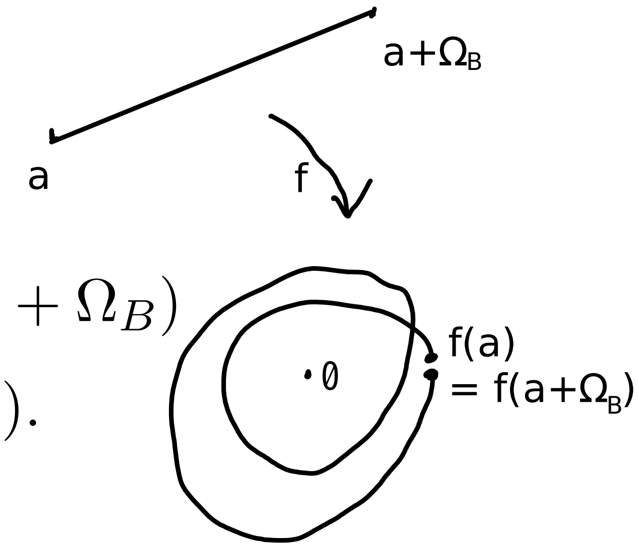
$$\begin{aligned}\int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} u\frac{f'(u)}{f(u)}\,du &= \int_a^{a+\Omega_B} (u+\Omega_A)\frac{f'(u+\Omega_A)}{f(u+\Omega_A)}\,du \\ &= -\int_{a+\Omega_B}^{a} u\frac{f'(u)}{f(u)}\,du - \Omega_A\int_{a+\Omega_B}^{a}\frac{f'(u)}{f(u)}\,du.\end{aligned}$$

Recall the proof of the "argument principle" on the winding number:

$$\int_{a+\Omega_B}^{a}\frac{f'(u)}{f(u)}\,du = \int_{a+\Omega_B}^{a} d\log f(u) = \log f(a) - \log f(a+\Omega_B).$$

Note that $f(a) = f(a+\Omega_B)$, BUT $\log f(a) \neq \log f(a+\Omega_B)$,

because of multivaluedness of $\log$.



$$\log f(a) = \log |f(a)| + i \arg f(a),$$

$$\log f(a+\Omega_B) = \log |f(a+\Omega_B)| + i \arg f(a+\Omega_B)$$
$$= \log |f(a)| + i \arg f(a+\Omega_B).$$

The argument is determined only up to $2\pi\mathbb{Z}$.

$$\log f(a+\Omega_B) - \log f(a) = i(\arg f(a+\Omega_B) - \arg f(a))$$
$$= 2\pi i n. \qquad (\exists n \in \mathbb{Z})$$

$$\implies \int_{a+\Omega_B}^{a} \frac{f'(u)}{f(u)}\,du = -2\pi i n.$$

$$\implies \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} u\frac{f'(u)}{f(u)}\,du = -\int_{a+\Omega_B}^{a} u\frac{f'(u)}{f(u)}\,du + 2\pi i n \Omega_A.$$

Similarly,

$$\begin{aligned}\int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} u\frac{f'(u)}{f(u)}\,du &= \int_{a+\Omega_A}^{a} (u+\Omega_B)\frac{f'(u+\Omega_B)}{f(u+\Omega_B)}\,du \\ &= -\int_{a}^{a+\Omega_A} u\frac{f'(u)}{f(u)}\,du - \Omega_B\int_{a}^{a+\Omega_A} \frac{f'(u)}{f(u)}\,du. \\ &= -\int_{a}^{a+\Omega_A} u\frac{f'(u)}{f(u)}\,du + 2\pi i m\Omega_B. \qquad (\exists m\in\mathbb{Z})\end{aligned}$$

Summing up,

$$\sum_{j=1}^{N}\alpha_j - \sum_{j=1}^{N}\beta_j = \frac{1}{2\pi i}\oint_{\partial\Pi} u\frac{f'(u)}{f(u)}\,du = n\Omega_A + m\Omega_B.$$

$\square$