Exercises in Invariant Theory 12.01.2021

These exercises do not have any due date before June. They have nothing to do with general theory we will develop in the class, but they provide some beautiful examples known in 19th century and described e.g. in Felix Klein's book on icosahedron.

1. Let $\pi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Galois cover of the Riemann sphere with coordinate z by another Riemann sphere with coordinate x, ramified over $z = 0, 1, \infty$ with ramification indices ν_0, ν_1, ν_∞, with Galois group Γ. Prove that
a) $\Gamma_3 = \mathfrak{A}_4$ (tetrahedron) has $(\nu_0, \nu_1, \nu_\infty) = (3, 2, 3)$;
b) $\Gamma_4 = \mathfrak{S}_4$ (octahedron) has $(\nu_0, \nu_1, \nu_\infty) = (3, 2, 4)$;
c) $\Gamma_5 = \mathfrak{A}_5$ (icosahedron) has $(\nu_0, \nu_1, \nu_\infty) = (3, 2, 5)$;
d) $\Gamma = \mathfrak{I}_n$ (dihedron) has $(\nu_0, \nu_1, \nu_\infty) = (2, 2, n)$;
e) There are no more finite subgroups in $SL(2, \mathbb{C})$, apart from the cyclic ones and the preimages of the above groups under $SL(2, \mathbb{C}) \to PGL(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1, x)$.

2. Prove that for $N = 3, 4, 5$ we have $\Gamma_N = \mathrm{PSL}(2, \mathbb{Z}/N\mathbb{Z})$, and the dihedral group $\mathfrak{I}_3 = \mathrm{PSL}(2, \mathbb{Z}/2\mathbb{Z})$.

3. Let $[x_1 : x_2]$ be the homogeneous coordinates on $(\mathbb{P}^1, x)$, and the preimage $\widetilde{\Gamma} \subset \mathrm{SL}(2, \mathbb{C})$ of $\Gamma \subset \mathrm{PGL}(2, \mathbb{C})$ acts linearly on x_1, x_2. Prove that the ring of $\widetilde{\Gamma}$-invariant polynomial functions $\mathbb{C}[x_1, x_2]^{\widetilde{\Gamma}}$ is generated by the following functions with the following relations
a) For Γ_5: $F_\infty = x_1x_2(x_1^{10} + 11x_1^5x_2^5 - x_2^{10})$,
$$F_0 = 228(x_1^{15}x_2^5 - x_1^5x_2^{15}) - (x_1^{20} + x_2^{20}) - 494x_1^{10}x_2^{10},$$
$$F_1 = (x_1^{30} + x_2^{30}) + 522(x_1^{25}x_2^5 - x_1^5x_2^{25}) - 1005(x_1^{20}x_2^{10} + x_1^{10}x_2^{20}),\ F_1^2 + F_0^3 = 1728F_\infty^5;$$
b) For Γ_4: $F_\infty^2,\ F_1F_\infty,\ F_0$, where $F_\infty = x_1x_2(x_1^4 - x_2^4),\ F_0 = x_1^8 + 14x_1^4x_2^4 + x_2^8$,
$$F_1 = x_1^{12} - 33x_1^8x_2^4 - 33x_1^4x_2^8 + x_2^{12} \text{ and } F_\infty^2(F_0^3 - 108F_\infty^4) = (F_1F_\infty)^2;$$
c) For Γ_3: $F_0^3,\ F_0F_\infty,\ F_1$, where $F_1 = x_1x_2(x_1^4 - x_2^4),\ F_0 = x_1^4 + 2\sqrt{-3}x_1^2x_2^2 + x_2^4$,
$$F_\infty = x_1^4 - 2\sqrt{-3}x_1^2x_2^2 + x_2^4 \text{ and } F_0^3(F_0^3 - 12\sqrt{-3}F_1^2) = (F_0F_\infty)^3;$$
d) For dihedral group for even n: $F_\infty^2,\ F_1^2,\ F_1F_0F_\infty$ and $(F_1F_0F_\infty)^2 = F_1^2F_\infty^2(F_1^2 - F_\infty^n)$, where $F_1 = (x_1^n + x_2^n)/2,\ F_0 = (x_1^n - x_2^n)/2,\ F_\infty = x_1x_2$;
for odd n: $F_\infty^2,\ F_1^2F_\infty,\ F_1F_0$ and $(F_1F_0)^2F_\infty^2 = (F_1^2F_\infty)(F_1^2F_\infty - F_\infty^{n+1})$;
e) In all these cases $z = \mathrm{const} \cdot F_0^{\nu_0}/F_\infty^{\nu_\infty},\ F_1 = \mathrm{const} \cdot \{F_0, F_\infty\} := \mathrm{const} \cdot \left(\frac{\partial F_0}{\partial x_1}\frac{\partial F_\infty}{\partial x_2} - \frac{\partial F_\infty}{\partial x_1}\frac{\partial F_0}{\partial x_2}\right)$ (Poisson bracket), and for $X := F_0F_\infty/F_1$ we have $x_2 = \sqrt{X(x_1, x_2)/X(x, 1)},\ x_1 = xx_2$;
f) Also, in cases a,b,c) we have $F_0 = \mathrm{const} \cdot \det \begin{pmatrix} \frac{\partial^2 F_\infty}{\partial x_1^2} & \frac{\partial^2 F_\infty}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_\infty}{\partial x_2 \partial x_1} & \frac{\partial^2 F_\infty}{\partial x_2^2} \end{pmatrix}$ (Hessian).

Exercises in Invariant Theory 19.01.2021

These exercises are due by January 26th. This is a general rule: the due date is one week after the assignment. The final grade for the course is calculated as 0.1 of the percentage of completely solved exercises. You may submit e.g. the high quality scans of your handwritten solutions in the natural order. I will grade neither poor quality scans nor randomly ordered scans.

1. For every positive integer n, give an example of a group Γ of order n and its representation V such that $\mathbb{C}[V]^G$ is *not* generated by invariants of degree less than n.

2. Let $\Gamma \subset GL(V)$ be a finite subgroup. Prove that
a) $\dim_{\mathbb{C}(V)^\Gamma} \mathbb{C}(V) = \sharp\Gamma$.
b) $\mathbb{C}(V) = \mathbb{C}(V)^\Gamma \otimes_{\mathbb{C}[V]^\Gamma} \mathbb{C}[V]$.
c) $\mathbb{C}(V)^\Gamma = \mathrm{Frac}(\mathbb{C}[V]^\Gamma)$.
d) $\dim_{\mathrm{Frac}(\mathbb{C}[V]^\Gamma)} \mathrm{Frac}(\mathbb{C}[V]^\Gamma) \otimes_{\mathbb{C}[V]^\Gamma} \mathbb{C}[V] = \sharp\Gamma$.

3. a) Prove that the *wreath product* $\mathfrak{S}_n \wr \mathbb{Z}/2\mathbb{Z} := \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ (the Weyl group of type B_n) in its natural n-dimensional representation V is a reflection group.
b) Find the generators of invariants in $\mathbb{C}[V]$ (in particular, their degrees).

4. The Weyl group of type B_n has an evident "sum" homomorphism onto $\mathbb{Z}/2\mathbb{Z}$. Its kernel K is the Weyl group of type D_n.
a) Prove that K in its n-dimensional representation V is a reflection group.
b) Find the generators of $\mathbb{C}[V]^K$ (in particular, their degrees).

5. More generally, $G(a,1,n) := \mathfrak{S}_n \wr \sqrt[a]{1}$ has an evident "product" homomorphism φ to the cyclic group $\sqrt[a]{1}$ of roots of unity. For a divisor p of a we define $G(a,p,n) \subset G(a,1,n)$ as the subgroup consisting of all g such that $\varphi(g)^{a/p} = 1$ (so that $\sharp G(a,p,n) = a^n n!/p$). The group $G(a,p,n)$ has a natural n-dimensional representation V.
a) Prove that $G(a,p,n)$ is a complex reflection group in V.
b) Find the generators of $\mathbb{C}[V]^{G(a,p,n)}$ (in particular, their degrees).
According to the Shephard-Todd classification, apart from the infinite series $G(a,p,n)$, there are 34 exceptional irreducible reflection groups (e.g. the group of symmetries of icosahedron).

Exercises in Invariant Theory 26.01.2021

1. a) Prove that every adjoint orbit of $GL(n)$ in $\mathfrak{gl}(n)$ has dimension at most $n^2 - n$. The orbits (and their elements) of dimension $n^2 - n$ are called *regular*.
b) Prove that a matrix $A \in \mathfrak{gl}(n) = \mathrm{Mat}_{n\times n}$ is regular iff its centralizer $\mathfrak{z}_{\mathfrak{gl}(n)}A$ coincides with the associative commutative subalgebra in $\mathrm{Mat}_{n\times n}$ generated by A. In other words, the minimal polynomial of A coincides with the characteristic polynomial of A.
c) Find the Jordan normal forms of regular matrices.

d) Prove that any point of the categorical quotient $\mathfrak{gl}(n)/\!\!/GL(n)$ has a unique (up to conjugation) regular representative in $\mathfrak{gl}(n)$.

2. Prove that the stabilizer (i.e. the centralizer) of a regular element is a commutative subgroup of $GL(n)$.

3. Choose a basis $v_1,\ldots,v_n$ of $V=\mathbb{C}^n$, and consider nilpotent operators $fv_i=iv_{i+1}$, $ev_i=(n-i+1)v_{i-1}$ and a semisimple operator $hv_i=n+1-2i$ (a *principal* $\mathfrak{sl}_2$*-triple* $\langle e,h,f\rangle$). Consider the *Kostant slice* $\Sigma=e+\mathfrak{z}_{\mathfrak{gl}(n)}f$. Prove
a) Σ consists of regular matrices.
b) The projection of Σ to $\mathfrak{gl}(n)/\!\!/GL(n)$ is an isomorphism.

4. Let c be the matrix of the cyclic permutation $v_i\mapsto v_{i-1},\ v_1\mapsto(-1)^{n-1}v_n$ (a *Coxeter element*), and let $U_\pm\subset SL(n)$ be the subgroup of strictly upper (resp. lower) triangular matrices. The *Steinberg cross-section* $\mathit{\Sigma}:=U_-c\cap cU_+$. Prove that
a) $\mathit{\Sigma}$ consists of the matrices with 1's just above the main diagonal, $(-1)^{n-1}$ in the bottom left corner, arbitrary entries elsewhere in the last row, and zeros everywhere else.
b) $\mathit{\Sigma}$ consists of regular matrices.
c) The projection of $\mathit{\Sigma}$ to $SL(n)/\!\!/SL(n)$ (conjugation action) is an isomorphism.

Exercises in Invariant Theory 02.02.2021

We consider the Grassmannian $\mathrm{Gr}(d,E)$ of codimension d subspaces in $E=\mathbb{C}^m$. For $V\in\mathrm{Gr}(d,E)$, the kernel of the natural projection $\Lambda^dE\to\Lambda^d(E/V)$ is a hyperplane in Λ^dE, i.e. a point in $\mathbb{P}(\Lambda^dE)^\vee$. Our goal is to describe the image of $\mathrm{Gr}(d,E)\hookrightarrow\mathbb{P}(\Lambda^dE)^\vee$ as a projective variety cut out by the *Plücker relations*.

The homogeneous functions on $(\Lambda^dE)^\vee$ are generated by the linear forms $v_1\wedge\ldots\wedge v_d$ for $v_i\in E$. Choosing a basis $e_1,\ldots,e_m\in E$ and a basis in $\mathbb{C}^d$, we can view $V\subset E$ as the kernel of a $d\times m$-matrix $A\colon\mathbb{C}^m\to\mathbb{C}^d$ of rank d. Then $\Lambda^dA\colon\Lambda^d\mathbb{C}^m\to\Lambda^d\mathbb{C}^d=\mathbb{C}$ takes $e_{i_1}\wedge\ldots\wedge e_{i_d}$ to the maximal minor of A with the columns $i_1,\ldots,i_d$. Its value $X_{i_1,\ldots,i_d}$ is the corresponding Plücker coordinate.

The Plücker relations are

$$X_{i_1,\ldots,i_d}\cdot X_{j_1,\ldots,j_d}-\sum\pm X_{i'_1,\ldots,i'_d}\cdot X_{j'_1,\ldots,j'_d},\tag{1}$$

where the sum runs over all the shuffles of $(i_1,\ldots,i_d,j_1,\ldots,j_d)$ permuting $j_1,\ldots,j_k$ with i's (preserving the orders of i's and j's), and $\pm$ is the sign of the shuffle permutation. Here we fix $0<k<d$, so that there are $d-1$ types of the above Plücker relations. More invariantly, the Plücker relations read

$$(v_1\wedge\ldots\wedge v_d)\cdot(w_1\wedge\ldots\wedge w_d)-\sum_{i_1<\ldots<i_k}(v_1\wedge\ldots\wedge w_1\wedge\ldots\wedge w_k\wedge\ldots\wedge v_d)\cdot(v_{i_1}\wedge\ldots\wedge v_{i_k}\wedge w_{k+1}\wedge\ldots\wedge w_d)=0$$

for any vectors $v_1,\ldots,v_d,w_1,\ldots,w_d\in E$.

1. *(Sylvester Lemma)* Prove that for $d \times d$ matrices M, N and $1 \le k \le d$ we have $\det M \cdot \det N = \sum \det M' \cdot \det N'$, where the sum runs over all the pairs of matrices M', N' obtained from M, N by shuffling the first k columns of N with some k columns of M.

2. Prove that a) Plücker relations (1) are satisfied for the Plücker coordinates of any subspace $V \in \mathrm{Gr}(d, E)$.

b) Conversely, given a point $(X_{i_1,\dots,i_d})$ satisfying the Plücker relations, the kernel of the matrix A with matrix elements $a_{s,t} = X_{i_1,\dots,i_{s-1},t,i_{s+1},\dots,i_d}$, $1 \le s \le d$, $1 \le t \le m$, is a codimension d subspace $V \subset E$ whose Plücker coordinates coincide with the given $(X_{i_1,\dots,i_d})$. Here we have chosen a collection $i_1, \dots, i_d$ such that $X_{i_1,\dots,i_d} \neq 0$ and rescaled the homogeneous coordinates so that $X_{i_1,\dots,i_d} = 1$.

c) The map $\mathrm{Gr}(d, E) \hookrightarrow \mathbb{P}(\Lambda^d E)^\vee$ is injective.

Thus by the Hilbert Nulstellensatz, $\mathrm{Gr}(d, E) \subset \mathbb{P}(\Lambda^d E)^\vee$ is cut out by the radical of the homogeneous ideal generated by the Plücker relations. In fact, this ideal coincides with its radical.

Our next goal is to describe the image of the flag variety $\mathcal{B} = \{E \supset V^1 \supset \dots \supset V^{m-1} \supset 0\}$ in its Plücker embedding $\mathcal{B} \hookrightarrow \prod_{d=1}^{m-1} \mathbb{P}(\Lambda^d E)^\vee$. We choose $m > d > r > 0$. Then the products of the Plücker coordinates $X_{i_1,\dots,i_d} \cdot X_{j_1,\dots,j_r}$ are bihomogeneous coordinates on $\mathbb{P}(\Lambda^d E)^\vee \times \mathbb{P}(\Lambda^r E)^\vee$ (in its Segre embedding).

The Plücker relations are

$$X_{i_1,\dots,i_d} \cdot X_{j_1,\dots,j_r} - \sum \pm X_{i'_1,\dots,i'_d} \cdot X_{j'_1,\dots,j'_r}, \tag{2}$$

where the sum runs over all the shuffles of $(i_1, \dots, i_d, j_1, \dots, j_r)$ permuting $j_1, \dots, j_k$ with i's (preserving the orders of i's and j's), and $\pm$ is the sign of the shuffle permutation. Here we fix $0 < k \le r$, so that there are r types of the above Plücker relations. More invariantly, the Plücker relations read

$$(v_1 \wedge \dots \wedge v_d) \cdot (w_1 \wedge \dots \wedge w_r) - \sum_{i_1 < \dots < i_k} (v_1 \wedge \dots \wedge w_1 \wedge \dots \wedge w_k \wedge \dots \wedge v_d) \cdot (v_{i_1} \wedge \dots \wedge v_{i_k} \wedge w_{k+1} \wedge \dots \wedge w_r) = 0$$

for any vectors $v_1, \dots, v_d, w_1, \dots, w_r \in E$.

3. Prove that Plücker relations (2) are satisfied for the Plücker coordinates of the *incidence subvariety* $\{V^d \subset V^r\} \subset \mathrm{Gr}(d, E) \times \mathrm{Gr}(r, E) \subset \mathbb{P}(\Lambda^d E)^\vee \times \mathbb{P}(\Lambda^r E)^\vee$.

4. Prove that given a point in $\prod_{d=1}^{m-1} \mathbb{P}(\Lambda^d E)^\vee$ satisfying Plücker relations (1,2) (for any pair $m > d > r > 0$), there is a flag $E \supset V^1 \supset \dots \supset V^{m-1} \supset 0$ with such coordinates.

Thus by the Hilbert Nulstellensatz, $\mathcal{B} \subset \prod_{d=1}^{m-1} \mathbb{P}(\Lambda^d E)^\vee$ is cut out by the radical of the homogeneous ideal generated by the Plücker relations. In fact, this ideal coincides with its radical.

Exercises in Invariant Theory 09.02.2021

1. Let $G \subset GL(V)$ be an algebraic subgroup. Suppose that the G-orbits in V are separated by G-invariant polynomials. Prove that G is finite.

2. Give an example of a finite solvable subgroup of $GL(2, \mathbb{C})$ not contained in any Borel subgroup.

3. Let G be a connected algebraic group, and let $N \subset G$ be a finite normal subgroup. Prove that N lies in the center of G.

4. Let $G_1 = \mathbb{G}_a^n$, and $G_2^0 = \mathbb{G}_m^r$ (connected component). Prove that there are neither nontrivial homomorphisms from G_1 to G_2, nor nontrivial homomorphisms from G_2 to G_1.

5. Let G be a connected nilpotent linear algebraic group, and let $H \subsetneq G$ be a closed connected subgroup. Prove that $\dim H < \dim N_G(H)$ (the normalizer). In particular, if $\dim H = \dim G - 1$, then H is normal.

Exercises in Invariant Theory 16.02.2021

1. Let $G = GL_n \supset T$ =diagonal matrices. For $1 \leq i \neq j \leq n$, consider a character $\alpha_{ij} \in X^*(T)\colon (t_1, \ldots, t_n) \mapsto t_i t_j^{-1}$. Prove that
a) α_{ij} is a root of $R(G, T)$.
b) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\alpha_{ij}}$.
c) G is reductive.
d) The root data $\Psi(G, T)$ are isomorphic to $(X, R, X^\vee, R^\vee)$, where $X = X^\vee = \mathbb{Z}^n$ with the standard pairing and basis $\{\varepsilon_i\}_{1 \leq i \leq n}$, $R = R^\vee = \{\varepsilon_i - \varepsilon_j\}_{i \neq j}$.
e) If $G_1 = SL_n \subset G$, $T_1 = T \cap G_1$, then $\Psi(G_1, T_1) \simeq (X_1, R_1, X_1^\vee, R_1^\vee)$, where $X_1 = X/(\varepsilon_1 + \ldots + \varepsilon_n)$, $X_1^\vee = \{(x_1, \ldots, x_n) \in X^\vee\colon x_1 + \ldots + x_n = 0\}$, $R_1^\vee = R^\vee \subset X_1^\vee$, and $R \subset X_1$ is the image of $R \subset X$ under the natural projection $X \twoheadrightarrow X_1$.
f) If $G' = PGL_n \twoheadleftarrow G$, T' is the image of T, then $\Psi(G', T') \simeq (X_1^\vee, R_1^\vee, X_1, R_1)$ (the first example of Langlands duality).
g) The Weyl group is isomorphic to the symmetric group $\mathfrak{S}_n$.
h) Choose a positive root subsystem R_1^+ in e) above, and find the corresponding simple roots, simple coroots and fundamental weights.

2. Let $V = \mathbb{C}^{2n+1}$ with symmetric bilinear form

$$((\xi_0, \ldots, \xi_{2n}), (\eta_0, \ldots, \eta_{2n})) = \xi_0\eta_0 + \sum_{i=1}^{n} (\xi_i \eta_{n+i} + \xi_{n+i}\eta_i).$$

The group $G = SO_{2n+1}$ consists of linear transformations of determinant 1, preserving this form. Prove that
a) $\mathfrak{g} = \mathfrak{so}_{2n+1}$ consists of linear transformations such that $(Av, w) + (v, Aw) = 0$.
b) A maximal torus T consists of transformations
$(\xi_0, \ldots, \xi_{2n}) \mapsto (\xi_0, t_1\xi_1, \ldots, t_n\xi_n, t_1^{-1}\xi_{n+1}, \ldots, t_n^{-1}\xi_{2n})$.

c) The characters $(t_1,\dots,t_n)\mapsto t_i^{\pm1}$ and $(t_1,\dots,t_n)\mapsto t_i^{\pm1}t_j^{\pm1},\ i\neq j$ are the roots of $R(G,T)$.

d) $\mathfrak{g}=\mathfrak{t}\oplus\bigoplus_\alpha \mathfrak{g}_\alpha$, where α runs through the set of roots of part c).

e) The root data $\Psi(G,T)$ are $(X,R,X^\vee,R^\vee)$, where $X=X^\vee=\mathbb{Z}^n$ with the standard pairing and basis $\{\varepsilon_i\}_{1\le i\le n},\ R=\{\pm\varepsilon_i,\ \pm\varepsilon_i\pm\varepsilon_j\ (i\neq j)\},\ R^\vee=\{\pm2\varepsilon_i,\ \pm\varepsilon_i\pm\varepsilon_j\ (i\neq j)\}$.

f) G is semisimple.

g) The Weyl group is isomorphic to the one of Problem 3a) of 19.01.

h) Choose a positive root subsystem R^+ in e) above, and find the corresponding simple roots, simple coroots and fundamental weights.

3. $V=\mathbb{C}^{2n}$ with symmetric bilinear form

$$((\xi_1,\dots,\xi_{2n}),(\eta_1,\dots,\eta_{2n}))=\sum_{i=1}^n(\xi_i\eta_{n+i}+\xi_{n+i}\eta_i).$$

The group $G=SO_{2n}$ consists of linear transformations of determinant 1, preserving this form. Prove that

a) A maximal torus T consists of transformations
$(\xi_1,\dots,\xi_{2n})\mapsto(t_1\xi_1,\dots,t_n\xi_n,t_1^{-1}\xi_{n+1},\dots,t_n^{-1}\xi_{2n})$.

b) The root data $\Psi(G,T)$ are isomorphic to $(X,R,X^\vee,R^\vee)$, where $X=X^\vee=\mathbb{Z}^n$ with the standard pairing and basis $\{\varepsilon_i\}_{1\le i\le n},\ R=\{\pm\varepsilon_i\pm\varepsilon_j\ (i\neq j)\}=R^\vee$ (an example of Langlands selfduality).

c) G is semisimple.

d) The Weyl group is isomorphic to the one of Problem 4 of 19.01.

e) Choose a positive root subsystem R^+ in b) above, and find the corresponding simple roots, simple coroots and fundamental weights.

4. $V=\mathbb{C}^{2n}$ with skew symmetric bilinear form

$$((\xi_1,\dots,\xi_{2n}),(\eta_1,\dots,\eta_{2n}))=\sum_{i=1}^n(\xi_i\eta_{n+i}-\xi_{n+i}\eta_i).$$

The group $G=Sp_{2n}$ consists of linear transformations preserving this form, and the torus $T\subset G$ is the same as in Problem 3. Prove that

a) T is a maximal torus, and G is semisimple.

b) The root data $\Psi(G,T)$ are isomorphic to $(X^\vee,R^\vee,X,R)$, where $(X,R,X^\vee,R^\vee)$ are as in Problem 2e) (another example of Langlands duality).

c) The Weyl group is isomorphic to the one of Problem 3a) of 19.01.

d) Choose a positive root subsystem $R^{\vee+}$ in b) above, and find the corresponding simple roots, simple coroots and fundamental weights.

These were the classical root systems of types A_{n-1},B_n,D_n,C_n.

5. In problems 2,3,4 let us consider a self-orthogonal flag $V=V^0\supset V^1\supset\ldots\supset V^m\supset 0$, where $m=2n$ in Problem 2, while $m=2n-1$ in Problems 3,4; and V^k stands for a vector subspace of codimension k in V; and the subspaces of complementary dimensions are

orthogonal with respect to the bilinear form in question. Prove that the stabilizer of this flag is a Borel subgroup.

Exercises in Invariant Theory 23.02.2021

1. Prove that any orbit of a unipotent group acting on an affine variety is closed.

2. Prove that a) $x \in \mathfrak{gl}_n$ is semisimple if and only if its adjoint orbit is closed.
b) Moreover, in this case the centralizer $Z_{GL_n}(x)$ is reductive.

3. Let G be a semisimple algebraic group, and $x \in \mathfrak{g} = \operatorname{Lie} G$. Assume that the centralizer $Z_G(x)$ is reductive. Prove that x is semisimple.

4. Prove that a) the nilpotent O_n-orbits in $\mathfrak{so}_n$ and the nilpotent Sp_{2n}-orbits in $\mathfrak{sp}_{2n}$ are uniquely determined by their Jordan types, i.e. partitions of n (resp. $2n$).
b) The partitions appearing for $\mathfrak{so}_n$ (resp. $\mathfrak{sp}_{2n}$) are precisely those where the multiplicity of every even (resp. odd) part is even.
c) Show that a nilpotent O_n-orbit splits into two SO_n-orbits if and only if the parts of the corresponding partition are all even.

5. Consider the additive group U formed by all the strictly upper triangular matrices $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $u \in \mathbb{C}$, acting on $\mathrm{Mat}_{2\times 2}$ by the left multiplications. Prove that there is *no* categorical quotient $\mathrm{Mat}_{2\times 2} /\!\!/ U$.

Exercises in Invariant Theory 02.03.2021

1. Let (e, h, f) be an $\mathfrak{sl}_2$-triple in a semisimple Lie algebra $\mathfrak{g}$. Then the eigenvalues of the adjoint action of h on $\mathfrak{g}$ are integral, so h is conjugate to a unique element of a dominant Weyl chamber C of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. That is, we choose a Borel subalgebra $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, it gives rise to a basis of simple positive roots $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R^+$, and $C = \{x \in \mathfrak{t} \,:\, \langle x, \alpha_i \rangle \geq 0 \;\forall\, i = 1, \dots, r\}$. So we will assume $h \in C$, and then $h = \sum_{i=1}^r d_i \omega_i^\vee$ for $d_i \in \mathbb{N}$ (here $\omega_1^\vee, \dots, \omega_r^\vee$ is the basis of fundamental coweights dual to the basis of simple roots). Prove that d_i must lie in $\{0, 1, 2\}$. In particular, the number of nilpotent orbits in $\mathfrak{g}$ is at most 3^r.

2. Let $\mathcal{B} = G/B$ be the flag variety of a connected semisimple group G (it parametrizes the Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$). Prove that the cotangent bundle $T^*\mathcal{B}$ is the variety of pairs $\{(\mathfrak{b}, x \in \operatorname{rad}_u \mathfrak{b})\}$ (a Borel subalgebra plus a nilpotent element contained in it). In particular, we have a surjective projection $\boldsymbol{\mu}\colon T^*\mathcal{B} \to \mathcal{N} \subset \mathfrak{g}$, where $\mathcal{N} \subset \mathfrak{g}$ is the subvariety formed by all the nilpotent elements (*Springer resolution*). In particular, the nilpotent cone $\mathcal{N}$ is irreducible and contains a unique open nilpotent orbit (*regular* or *principal* orbit).

3. Let $\mathfrak{n}$ be the nilpotent radical of a Borel subalgebra $\mathfrak{b}$. Prove that
a) $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{i=1}^r \mathfrak{g}_{\alpha_i}$ (the positive simple root spaces).

b) An element $e \in \mathfrak{n}$ is regular iff its projection to $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is of the form $\sum_{i=1}^r c_i e_{\alpha_i}$ for $c_i \neq 0$ (for any i).

c) The corresponding h (to a regular e above) is conjugate to $\sum_{i=1}^r 2\omega_i^\vee$ (in notation of Problem 1).

4. Prove that a) for a nilpotent $e \in \mathfrak{g}$ the dimension of the centralizer $\mathfrak{z}_\mathfrak{g}(e)$ is at least the rank r of $\mathfrak{g}$;

b) If $\dim \mathfrak{z}_\mathfrak{g}(e) = r$, then e is regular (principal).

5. According to B. Kostant, given a nilpotent $f \in \mathfrak{gl}(V)$, any two $\mathfrak{sl}_2$-triples with this f are conjugate. Prove that there is a unique filtration $0 = V_{-N-1} \subset V_{-N} \subset \ldots \subset V_{N-1} \subset V_N = V$ such that $fV_i \subset V_{i-2}$, and f^i induces an isomorphism of the associated graded pieces from $\mathrm{gr}_i V$ to $\mathrm{gr}_{-i} V$ for any $i \geq 0$ (where $\mathrm{gr}_i V := V_i/V_{i-1}$). It is called the *monodromy filtration* for reasons that will become clear from Vologodsky course on Weil conjectures. In fact, such a filtration is defined for any object V of an abelian category equipped with a nilpotent endomorphism f.

Exercises in Invariant Theory 09.03.2021

1. The algebra $\mathbb{C}[\mathfrak{gl}_n]^{GL_n}$ is freely generated by the invariant functions $\mathrm{Tr}\, A, \ldots, \mathrm{Tr}\, A^n$. Write down the generators of $\mathbb{C}[\mathfrak{so}_n]^{SO_n}$ and $\mathbb{C}[\mathfrak{sp}_{2n}]^{Sp_{2n}}$ corresponding to the generators of $\mathbb{C}[\mathfrak{t}]^W$ you chose in Problems 3b), 4b) of January 19th.

2. Let a torus T act on a vector space V with an eigen-basis $v_1, \ldots, v_n$ with eigen-characters $\chi_1, \ldots, \chi_n$. Let $D \subset GL(V)$ be the maximal torus of all operators diagonal in the basis $v_1, \ldots, v_n$. We have $T \subset D$. Note that D/T is also a torus, and it acts naturally on $\mathbb{C}[V]^T$, and the character lattice $X^*(D/T)$ embeds naturally into $X^*(D) = \mathbb{Z}^n$. Let $\mathcal{M}$ denote the submonoid $\{(d_1, \ldots, d_n) \in \mathbb{N}^n \ : \ \sum_{i=1}^n d_i\chi_i = 0\} \subset \mathbb{N}^n$. Prove that

a) Each eigenspace of D/T in $\mathbb{C}[V]^T$ has dimension 0 or 1, and the dimension is 1 iff the corresponding eigencharacter lies in $\mathcal{M}$. For $\psi \in \mathcal{M}$ let f_ψ denote an eigenvector with eigenvalue ψ in $\mathbb{C}[V]^T$.

b) Characters $\psi_1, \ldots, \psi_k$ generate the monoid $\mathcal{M}$ iff the polynomials $f_{\psi_1}, \ldots, f_{\psi_k}$ generate the algebra $\mathbb{C}[V]^T$. Deduce from this statement that $\mathcal{M}$ is a finitely generated monoid.

c) Relations of the form $f_{\psi_1}^{a_1} \cdots f_{\psi_k}^{a_k} - f_{\psi_1}^{b_1} \cdots f_{\psi_k}^{b_k} = 0$ with $a_i, b_i \geq 0$ and $\sum_{i=1}^k (a_i - b_i)\psi_i = 0$, generate the ideal of relations between the generators $f_{\psi_1}, \ldots, f_{\psi_k}$.

3. a) Let $G = GL(U)$ act on $V = U^{\oplus k} \oplus U^{*\oplus \ell}$. Prove that $(u_1, \ldots, u_k, u^1, \ldots, u^\ell)$ lies in the nullcone iff $\langle u^j, u_i \rangle = 0$ for any i, j.

b) Let $G = SL(U)$ act on $U^{\oplus k}$. Prove that $(u_1, \ldots, u_k)$ lies in the nullcone iff $u_1, \ldots, u_k$ do not span U.

c) Let $G = O(U)$ (where U is equipped with a nondegenerate symmetric form $(,)$) act on $U^{\oplus k}$. Prove that $(u_1, \ldots, u_k)$ lies in the nullcone iff $(u_i, u_j) = 0$ for any i, j.

d) Find the nullcone for $G = Sp(U)$ acting on $U^{\oplus k}$.

You are assumed to use the Hilbert-Mumford theorem (as opposed to calculation of the algebra of invariant functions).

4. Consider the action of $G = SL(3)$ on $\mathrm{Sym}^3 V,\ V = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$. The eigenvectors of the diagonal torus of G are cubic monomials (e.g. xyz has weight 0). Prove that up to the action of the Weyl group (permutations of x, y, z), there is a unique maximal convex polygon with vertices in the weights of $\mathrm{Sym}^3 V$, not containing 0 (e.g. a triangle $\mathcal{W}$ with vertices in the weights of the monomials x^3, y^3, y^2z).

5. a) Prove that the support of a cubic form P is contained in $\mathcal{W}$ iff the point $(0, 0, 1) \in \mathbb{P}^2$ is either a triple point of the corresponding projective cubic $\{P = 0\} \subset \mathbb{P}^2$ or a double point with the tangent line $\{y = 0\}$ common to the two branches. Hence the nullcone consists of equations of plane cubics that have singularities worse than nodal.

b) Prove that the orbit of P is closed 8-dimensional iff the corresponding plane cubic is smooth.

Exercises in Invariant Theory 16.03.2021

1. Let A be a hermitian operator on $V \ni v,\ Av \neq 0,\ f(t) := |\exp(tA)v|^2$. Prove that
a) t_0 is a critical point of f iff $v_0 = \exp(t_0A)v$ satisfies $\langle Av_0, v_0\rangle = 0$.
b) There is at most one critical point of f.
c) Such a critical point is the point of global minimum.

2. Prove that $\boldsymbol{\mu}\colon T^*\mathcal{B} \to \mathfrak{g}$ of Problem 2 of March 2nd is a moment map of the Hamiltonian action $G \curvearrowright T^*\mathcal{B}$ (arising from the action $G \curvearrowright \mathcal{B}$) under the identification $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form.

3. A double quiver Q is a set I of vertices, a set H of arrows with a map $h \mapsto h', h'',\ H \to I \times I$ (tail and head), and an involution $h \mapsto \bar{h},\ H \to H$ such that $\bar{h}' = h'',\ \bar{h}'' = h'$. Given vector spaces $V_i,\ i \in I$, the space of representations $E_V = \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''})$. Choosing a function $\varepsilon\colon H \to \mathbb{C}^\times$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$ (e.g. a choice of orientation gives rise to $\varepsilon = \pm 1$) we obtain a symplectic form on the vector space $E_V\colon \langle (x_h), (y_h)\rangle = \sum_{h\in H} \varepsilon_h \mathrm{Tr}_{V_{h'}} x_h y_{\bar{h}}$. Prove that $\boldsymbol{\mu}\colon E_V \to \bigoplus_{i\in I} \mathrm{End}\, V_i = \bigoplus_{i\in I} \mathfrak{gl}(V_i) \cong \bigoplus_{i\in I} \mathfrak{gl}(V_i)^*,\ (x_h) \mapsto \sum_{h\in H} \varepsilon_h x_h x_{\bar{h}}$ is a moment map of the Hamiltonian action $\prod_{i\in I} GL(V_i) \curvearrowright E_V,\ (g_i)(x_h) = (g_{h''} x_h g_{h'}^{-1})$.

4. Let P be a homogeneous degree d polynomial in x, y. It lies in the nullcone with respect to the natural action of $SL(2)$ on $\mathrm{Sym}^d \mathbb{C}^2$ iff it has a linear factor of multiplicity $k > d/2$. In this case find a characteristic of P.

5. In the setup of Problems 4,5 of March 9th, find characteristics of the following cubic forms in the nullcone:
a) $P = y^2z - x^3$.
b) $P = xy(x - y)$.
c) $P = (x^2 + y^2 - 2yz)y$.

Exercises in Invariant Theory 23.03.2021

1. Prove that any parabolic subgroup of GL_n is the stabilizer of a (not necessarily complete) flag in $\mathbb{C}^n$.

2. Let G be a reductive algebraic group, and $\nu\colon \mathbb{G}_m \to G$ a homomorphism. For $g \in G$ we set $\theta_{g,\nu}\colon \mathbb{G}_m \to G,\ t \mapsto \nu(t)g\nu(t)^{-1}$. Let $P(\nu) \subset G$ be the subset formed by all $g \in G$ such that $\theta_{g,\nu}$ has a limit as $t \to 0$. Prove that

a) $P(\nu)$ is a subgroup of G containing the centralizer $H = Z_G(\nu(\mathbb{G}_m))$.

b) $P(g\nu g^{-1}) = gP(\nu)g^{-1}$.

c) $P(\nu) \cap P(-\nu) = H$.

d) If $G = GL(V)$, then for any ν there is a flag $V_1 \subset V_2 \subset \ldots \subset V_s \subset V$ (not necessarily complete) such that $P(\nu)$ is the stabilizer of this flag.

e) $P(\nu)$ is a parabolic subgroup of G for any ν (and any reductive G).

3. Let $G = GL(V)$. We say that $g \in G$ contracts $x \in G$ if $\lim\limits_{n\to\infty} g^n x g^{-n} = \mathrm{Id}$. We say that x is g-bounded if the closure of the set $\{g^n x g^{-n},\ n \in \mathbb{N}\}$ is compact (in the classical topology). For $g \in G$ let $U_g \subset G$ (resp. $P_g \subset G$) be the set of all g-contractible (resp. g-bounded) elements of G. Prove that

a) for any $g \in G$, U_g is a unipotent subgroup equal to the unipotent radical of a certain parabolic subgroup.

b) The normalizer of U_g is a parabolic subgroup.

c) P_g is contained in the normalizer of U_g and is equal to this normalizer iff g is semisimple.

4. Let $\mathcal{D}_n^+ \subset \mathrm{Mat}_{k\times k}^+$ (resp. $\mathcal{D}_n^- \subset \mathrm{Mat}_{k\times k}^-$) denote the scheme theoretic intersection of the determinantal scheme $\mathcal{D}_n$ of all matrices of rank at most n with the subspace $\mathrm{Mat}_{k\times k}^+$ of symmetric matrices (resp. subspace $\mathrm{Mat}_{k\times k}^-$ of skew-symmetric matrices). Assuming that the scheme $\mathcal{D}_n^\pm$ is reduced and normal, prove that $(\mathbb{C}^n)^{\oplus k}/\!\!/O_n \cong \mathcal{D}_n^+$ and $(\mathbb{C}^n)^{\oplus k}/\!\!/Sp_n \cong \mathcal{D}_n^-$.

5. Prove that $\mathrm{Mat}_{k\times k}^+/\!\!/O_k$ (the quotient with respect to conjugation action) is the polynomial algebra with generators $\mathrm{Tr}\,A, \ldots, \mathrm{Tr}\,A^k$.

Exercises in Invariant Theory 30.03.2021

1. Recall the setup of Problem 2 of March 9th. Pick a character θ of T. Prove that

a) $V^{\theta-ss} \subset V$ is D-invariant, and D/T acts on $V/\!\!/_\theta T$ in such a way that $\pi^\theta\colon V^{\theta-ss} \to V/\!\!/_\theta T$ is D-equivariant.

b) The fixed points of D/T on $V/\!\!/_\theta T$ are in bijection with the subsets $I \subset \{1, \ldots, n\}$ satisfying the following two conditions:

i) $\chi_i,\ i \in I$, are linearly independent;

ii) There are negative rational numbers $n_i,\ i \in I$, such that $\theta = \sum_i n_i \chi_i$.

2. Let G be a connected factorial reductive algebraic group (i.e. the algebra $\mathbb{C}[G]$ is a unique factorization domain), and let H be an algebraic subgroup of G. We have a natural

restriction morphism of character groups $\varrho\colon X^*(G) \to X^*(H)$. Prove that $\mathrm{Pic}(G/H) \cong \mathrm{Coker}\,\varrho$.

3. Let Y be a factorial affine algebraic variety, and let G be a connected linear algebraic group without characters: $X^*(G) = \{1\}$. Prove that

a) $\mathbb{C}[Y]^G$ is a unique factorization domain as well.

b) There are finitely many elements $f_1, \ldots, f_k \in \mathbb{C}[Y]^G$ and a Zariski open G-invariant subset $Y' \subset Y$ such that for $y_1, y_2 \in Y'$ the following conditions are equivalent:

i) $f_i(y_1) = f_i(y_2)$ for any $i = 1, \ldots, k$.

ii) $Gy_1 = Gy_2$.

4. Consider the action of $G = SL(2)$ in $\mathrm{Sym}^3 \mathbb{C}^2$ (cubic polynomials in x, y). Let $v = x^2y$, and $\Upsilon(t) = \begin{pmatrix} t^{-1} & 0 \\ -t^{-2} & t \end{pmatrix}$. Prove that

a) $Gx^3 \subset \overline{Gv}$.

b) The connected component $\mathrm{Stab}^0_G\, x^3$ of the stabilizer of x^3 is a 1-dimensional unipotent group.

c) For any Cartan torus $T \subset G$ we have $Gx^3 \cap \overline{Tv} = \emptyset$.

Hence a *non-closed* orbit Gx^3 in the closure of Gv cannot be reached as the limit of a one-parametric subgroup orbit of v.

5. Consider the action of $G = SL(2)$ in $V = \mathrm{Sym}^d \mathbb{C}^2$, $d \geq 3$. Let $v = x^d + y^d$. Let $H = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\}$, where $\zeta^d = 1$ (a finite cyclic subgroup). Prove that

a) The centralizer of H in G is the subgroup D of diagonal matrices.

b) $v \in V^H$ and the orbit Dv is closed.

c) The orbit Gv is closed (by the D. Luna criterion).

Exercises in Invariant Theory 06.04.2021

1. Consider the Grassmannian cone $C\mathrm{Gr}(m, k) \subset \Lambda^m \mathbb{C}^k$ of decomposable m-vectors (with the reduced scheme structure). Define $f \in \mathbb{C}[C\mathrm{Gr}(m, k)]$ that sends $\sum_I c_I v_{i_1} \wedge \ldots \wedge v_{i_m}$ to $c_{\{1,2,\ldots,m\}}$. Prove that $\mathbb{C}[C\mathrm{Gr}(m, k)]^U$ (where U is the group of strictly upper triangular matrices) is the polynomial algebra $\mathbb{C}[f]$. Hence $C\mathrm{Gr}(m, k)$ is a normal irreducible variety.

2. For a commutative algebra A equipped with a locally finite action of a reductive group G, and a dominant weight $\lambda \in X^*_+(T)$, we denote by A_λ the V_λ-isotypic component of A. We say that A is $X^*_+(T)$-graded if $A = \bigoplus_{\lambda \in X^*_+(T)} A_\lambda$ is an algebra grading.

a) Prove that the category of $X^*_+(T)$-graded G-algebras is equivalent to the category of $X^*_+(T)$-graded algebras via $A \mapsto A^U$ (where U is the unipotent radical of the Borel $B \supset T$), and the inverse functor is $B \mapsto \big(\mathbb{C}[G/U] \otimes B\big)^T$, where the action of T on $V_\mu \otimes A_\lambda$ is via the character $t \mapsto \mu(t)\lambda(t)^{-1}$.

b) For an arbitrary commutative G-algebra A define an increasing $\mathbb{N}$-filtration $A_i := \bigoplus_{\langle\lambda,\rho^\vee\rangle\leq i} A_\lambda$. Prove that this is an algebra filtration, and $\operatorname{gr} A = \left(\mathbb{C}[G/U] \otimes A^U\right)^T$. In particular, $\operatorname{gr} A$ is finitely generated if A is.

3. Let a reductive group G with a Cartan torus T act on a vector space V, and let $v_1, \ldots, v_n$ be T-eigenvectors with distinct eigencharacters $\chi_1, \ldots, \chi_n$. Assume that 0 is an inner point of the convex hull of $\chi_1, \ldots, \chi_n$ in $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, and also that $\chi_i - \chi_j$ is never a root of (G, T). Deduce from the Kempf-Ness theorem that the orbit $Gv \subset V$ is closed, where $v = v_1 + \ldots + v_n$.

4. $G = SL_2$ acts in the space $\operatorname{Sym}^d \mathbb{C}^2$ of degree d homogeneous polynomials in x, y. Let $P = \sum_{i=0}^d a_i x^i y^{d-i}$, where $a_i a_{i+1} = 0$ for any i, and $\max\{i : a_i \neq 0\} > d/2 > \min\{i : a_i \neq 0\}$. Prove that the orbit GP is closed.

5. Find a semisimple element of SO_n whose centralizer is not connected.

Exercises in Invariant Theory 13.04.2021

1. Prove that a) the orthogonal group O_n is not connected.

b) The Cayley transform $X \mapsto (1-X)(1+X)^{-1}$ gives an isomorphism between an open subset of SO_n and an open subset of $\mathfrak{so}_n$.

c) $SO_n = O_n^0$.

2. Let $X \subset \mathbb{A}^n$ be the hypersurface $\{x = (x_1, \ldots, x_n) : (x, x) = x_1^2 + \ldots + x_n^2 = 1\}$ and let $e = (0, \ldots, 0, 1)$. Prove that

a) SO_n acts on X, and the stabilizer of e is SO_{n-1}.

b) The categorical quotient $SO_n /\!\!/ SO_{n-1} \cong X$ (with respect to right translations).

c) SO_n is connected.

3. For $x \in X$ we define the reflection $s_x(y) = y - 2(y, x)x$; it is an element of O_n. Prove that

a) O_n is generated by the reflections s_x.

b) For $x_1, x_2 \in X$ we have $s_{x_1} s_{x_2} \in SO_n$.

c) SO_n is connected.

4. We consider $\mathbb{C}^{2n}$ with the standard symplectic form. For $x, y \in \mathbb{C}^{2n}$ and $a \in \mathbb{C}$ we define $t_{x,a}(y) = y - a\langle x, y\rangle x$. Prove that

a) Sp_{2n} is generated by the transformations $t_{x,a}$.

b) Sp_{2n} is connected.

5. Construct an isomorphism between

a) GL_n/O_n and an open subvariety of the space of symmetric $n \times n$ matrices.

b) GL_{2n}/Sp_{2n} and an open subvariety of the space of skew symmetric $2n \times 2n$ matrices.

Exercises in Invariant Theory 20.04.2021

1. Recall the setup of Problems 3,4 of March 2nd. Fix $e=\sum_{i=1}^r e_{\alpha_i}$, choose a principal $\mathfrak{sl}_2$-triple (e,h,f) containing e, and set $\Sigma:=e+\mathfrak{z}_\mathfrak{g}(f)$. Also set $\mathfrak{g}^1_*=\{\mathfrak{b}_-+\sum_{i=1}^r c_ie_{\alpha_i}\ :\ c_i\neq 0\}$. Prove that

a) $\mathrm{Ker}(\mathrm{ad}_e)\cap\mathfrak{b}_-=0$.

b) $\mathfrak{g}=\mathrm{Im}(\mathrm{ad}_e)\oplus\mathfrak{z}_\mathfrak{g}(f)$.

c) If $x\in\mathfrak{n}_-,\ p\in e+\mathfrak{b}_-$, and $[x,p]=0$, then $x=0$.

2. Prove that the adjoint action a) $U_-\times\Sigma\to e+\mathfrak{b}_-,\ (u,s)\mapsto\mathrm{Ad}_u\,s$, is an isomorphism.

b) $B_-\times\Sigma\to\mathfrak{g}^1_*,\ (b,s)\mapsto\mathrm{Ad}_b\,s$, is an isomorphism.

3. Prove that a) for a simple root $\alpha,\ f_{-\alpha}\in\mathfrak{g}_{-\alpha}$ such that $[f_{-\alpha},e]=-\alpha^\vee$, and $a\in\mathfrak{t}$ we have $s_\alpha(a)+e=\exp\big(\alpha(a)\cdot\mathrm{ad}_{f_{-\alpha}}\big)(a+e)$, where the simple reflection $s_\alpha(a)=a-\alpha(a)\alpha^\vee$.

b) The projection $\mathfrak{t}\to\mathfrak{g}^*_1/B_-\cong\Sigma,\ a\mapsto a+e$, factors through $\mathfrak{t}\to\mathfrak{t}/W\to\Sigma$.

4. Prove that a) the closure of the conjugacy class $\mathrm{Ad}_G(a+e)$ contains the conjugacy class $\mathrm{Ad}_G(a)$ for any $a\in\mathfrak{t}$.

b) The composition $\mathfrak{t}/W\to\mathfrak{g}^*_1/B_-\to\mathfrak{g}/\!\!/G\cong\mathfrak{t}/W$ is $\mathrm{Id}\colon\mathfrak{t}/W\to\mathfrak{t}/W$.

5. a) Prove that the morphisms $\mathfrak{t}/W\to\Sigma\to\mathfrak{g}/\!\!/G\cong\mathfrak{t}/W$ are the mutually inverse isomorphisms. (Recall that $\dim\mathfrak{z}_\mathfrak{g}(f)=\dim\mathfrak{t}$ since the eigenvalues of the adjoint action of h in $\mathfrak{g}$ are all even, and hence $\dim\mathfrak{g}_0=\dim\mathfrak{g}^f$).

b) The dilation action of $\mathbb{C}^\times$ on $\mathfrak{t}$ induces its action $(c,t)\mapsto c\cdot t$ on $\mathfrak{t}/W$. Let $H_c=\begin{pmatrix}c^{-1}&0\\0&c\end{pmatrix}\in SL_2$ with Lie algebra $\mathfrak{sl}_2=\langle e,h,f\rangle$. Prove that the action of $\mathbb{C}^\times$ on $\Sigma\cong\mathfrak{t}/W$ corresponding to $(c,t)\mapsto c^2\cdot t$ is $(c,z)\mapsto c^2\,\mathrm{Ad}_{H_c}\,z$ for $z\in e+\mathfrak{z}_\mathfrak{g}(f)$.

c) Prove that Σ consists of regular elements (whose centralizer has dimension equal to the rank of $\mathfrak{g}$).

This is an elegant proof of B. Kostant's theorems about Kostant slice due to A. Beilinson and V. Drinfeld.

6. a) The family of centralizers forms a vector bundle (*universal centralizer*) $\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$ over Σ. Prove that all the centralizers of regular elements are abelian Lie algebras, so $\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$ is a vector bundle of algebraic abelian Lie subalgebras of $\mathfrak{g}$.

b) Prove that the vector bundle $\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$ is isomorphic to the cotangent bundle $T^*\Sigma$.

7. Let $\mathfrak{g}^{\mathrm{rss}}\subset\mathfrak{g}^{\mathrm{reg}}\subset\mathfrak{g}$ be the subsets formed by all the regular semisimple (resp. regular) elements. Prove that

a) $\mathfrak{g}^{\mathrm{rss}}$ and $\mathfrak{g}^{\mathrm{reg}}$ are open.

b) The complement $\mathfrak{sl}_2\setminus\mathfrak{sl}_2^{\mathrm{reg}}$ has codimension 3 in $\mathfrak{sl}_2$.

c) The complement $\mathfrak{g}\setminus\mathfrak{g}^{\mathrm{reg}}$ has codimension at least 3 in $\mathfrak{g}$.

Exercises in Invariant Theory 27.04.2021

1. Recall that we have the commuting actions of $GL(V)$ and the symmetric group $\mathfrak{S}_d$ on $V^{\otimes d}$, and $\mathrm{End}_{GL(V)}(V^{\otimes d}) = \mathbb{C}[\mathfrak{S}_d]$ by the first fundamental theorem of invariant theory for $GL(V)$. Prove that $\mathrm{End}_{\mathfrak{S}_d}(V^{\otimes d})$ is spanned by the operators $\rho(g)\colon V^{\otimes d} \to V^{\otimes d},\ g \in GL(V)$.

2. Prove that a) for any $c \in \mathbb{C}[\mathfrak{S}_d]$ the natural map $c\mathbb{C}[\mathfrak{S}_d] \otimes_{\mathbb{C}[\mathfrak{S}_d]} V^{\otimes d} \to cV^{\otimes d}$ is an isomorphism of $GL(V)$-modules.

b) If $W = c\mathbb{C}[\mathfrak{S}_d]$ is an irreducible right $\mathfrak{S}_d$-module, then $W \otimes_{\mathbb{C}[\mathfrak{S}_d]} V^{\otimes d} \cong cV^{\otimes d}$ is an irreducible $GL(V)$-module.

c) If U is an irreducible left $\mathfrak{S}_d$-module, then $\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_d]}(U, V^{\otimes d})$ is an irreducible $GL(V)$-module.

d) As a module over $GL(V) \times \mathfrak{S}_d$, $V^{\otimes d}$ decomposes into direct sum of irreducibles $V^{\otimes d} = \bigoplus_{U_i \in \mathrm{Irr}(\mathfrak{S}_d)} S_i \otimes U_i$. Here S_i is an irreducible $GL(V)$-module, and $U_i \mapsto S_i$ is an embedding of $\mathrm{Irr}(\mathfrak{S}_d) \hookrightarrow \mathrm{Irr}(GL(V))$ (*Schur-Weyl reciprocity*).

3. Let $\dim V = n$. For a partition λ with $\le n$ parts let us denote by $S^\lambda V$ the corresponding irreducible $GL(V)$-module.

a) Prove that $S^\lambda V$ enters $V^{\otimes d}$ with nonzero multiplicity iff λ is a partition of d.

Thus the irreducible representations of $\mathfrak{S}_d$ are naturally numbered by partitions of d: $\lambda \mapsto U_\lambda$. Historically this was proved by G. Frobenius long before the classification of $GL(V)$-modules, and Frobenius theorem was used by I. Schur to classify $GL(V)$-modules. In the next assignment we will use two basic facts about $\mathfrak{S}_d$-modules:

b) Prove that $U_\lambda^* \simeq U_\lambda$.

c) Prove that $U_\lambda \otimes \mathrm{sign} \simeq U_{\lambda^t}$ (transposed Young diagram).

4. Let $\dim V = n \le m = \dim W$. We may identify the $GL(V) \times GL(W)$-module $\mathrm{Sym}^\bullet(V \otimes W)$ with $\mathbb{C}[\mathrm{Mat}_{n\times m}]$ with the action $(g,h)M = {}^t g^{-1} M h^{-1}$. Prove that

a) The upper triangular Borel subgroup of $GL(V) \times GL(W)$ has an open orbit $\mathbb{O}_0$ in $\mathrm{Mat}_{n\times m}$.

b) Any irreducible $GL(V) \times GL(W)$-module enters $\mathbb{C}[\mathrm{Mat}_{n\times m}]$ with multiplicity either 0 or 1.

5. a) Prove that for the diagonal tori $D_n \subset GL(V)$ and $D_m \subset GL(W)$ there is a point M in the open Borel orbit $\mathbb{O}_0$ with $\mathrm{Stab}_{D_n\times D_m} M = \{(t_1^{-1}, \ldots, t_n^{-1}); (t_1, \ldots, t_n, t_{n+1}, \ldots, t_m)\}$.

b) For a partition μ with $\le m$ parts let us denote by $S^\mu W$ the corresponding irreducible $GL(W)$-module. Prove that if $S^\lambda V \otimes S^\mu W$ enters $\mathbb{C}[\mathrm{Mat}_{n\times m}]$ (with multiplicity 1), then $\mu = \lambda$.

Exercises in Invariant Theory 04.05.2021

1. Prove that a) for $1 \le k \le n$ the upper left $k\times k$-minor of a matrix in $\mathrm{Mat}_{n\times m}$ is a highest vector of the $GL(V) \times GL(W)$-module $\mathbb{C}[\mathrm{Mat}_{n\times m}]$ of the fundamental weight (ω_k, ω_k).

b) For any partition λ with $\leq n$ parts the $GL(V)\times GL(W)$-module $S^\lambda V\otimes S^\lambda W$ enters $\mathbb{C}[\mathrm{Mat}_{n\times m}]$ with multiplicity 1.

Thus, $\mathrm{Sym}^d(V\otimes W)=\bigoplus_\lambda S^\lambda V\otimes S^\lambda W$, where the sum runs over all partitions of size d with $\leq n$ parts. This is the so called *Howe* (GL_n, GL_m)*-duality* (equivalent to the Cauchy identity for Schur functions, and known long before R. Howe. However the above proof is due to R. Howe).

2. Here is an alternative way to deduce the (GL_n, GL_m)-duality from the Schur-Weyl reciprocity. Note that

$$\mathrm{Sym}^d(V\otimes W)=(V^{\otimes d}\otimes W^{\otimes d})^{\Delta\mathfrak{S}_d}=\bigoplus_{\lambda,\mu} S^\lambda V\otimes S^\mu W\otimes (U_\lambda\otimes U_\mu)^{\Delta\mathfrak{S}_d}.$$

Prove that $(U_\lambda\otimes U_\mu)^{\Delta\mathfrak{S}_d}=\mathbb{C}^{\delta_{\lambda\mu}}$. The Howe duality follows.

3. Prove that $\Lambda^d(V\otimes W)=\bigoplus_\lambda S^\lambda V\otimes S^{\lambda^t}W$, where the sum runs over all partitions of size d whose Young diagram fits into the $n\times m$-box (and λ^t stands for the transposed Young diagram). This is the so called *skew Howe duality*.

4. Prove that a) $\mathrm{End}_{GL(V)}(\mathrm{Sym}^d(V\otimes W))$ is spanned by the operators $\rho(g)\colon \mathrm{Sym}^d(V\otimes W)\to \mathrm{Sym}^d(V\otimes W),\ g\in GL(W)$.

b) $\mathrm{End}_{GL(V)}(\Lambda^d(V\otimes W))$ is spanned by the operators $\rho(g)\colon \Lambda^d(V\otimes W)\to\Lambda^d(V\otimes W),\ g\in GL(W)$.

5. Recall the setup of Problem 4 of March 30. Prove that

a) $\mathrm{Stab}_G\, v$ is trivial.

b) For a Zariski open subset $U\subset \mathrm{Sym}^3\mathbb{C}^2$ and any point $u\in U$ the stabilizer $\mathrm{Stab}_G\, u$ is isomorphic to the group of cubic roots of unity.

Thus the (nonclosed) orbit Gv does *not* admit a slice in the Zariski topology (and even in the étale topology).

c) Prove that v does *not* have any G-invariant affine neighbourhood in the open subvariety $Y\subset\mathrm{Sym}^3\mathbb{C}^2$ equal to the union of all 3-dimensional G-orbits.

Exercises in Invariant Theory 11.05.2021

1. Let $X=GL_n/SO_n$, be a two-fold covering of the space of nondegenerate quadratic forms, and let $\pi\colon GL_n\to X$ be the natural projection.

a) Prove that there is no rational section $s\colon X\dashrightarrow GL_n$ of π (i.e. no Zariski open $Y\subset X$ with a section $s\colon Y\to GL_n$ of $\pi|_{\pi^{-1}(Y)}$) when $n>2$. Hence the action of SO_n on GL_n admits no slice in Zariski topology.

b) Construct a rational section for $n=2$.

2. Let $G=GL(V)$ act naturally on $E=\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)$, and let $M=\{(p,q)\in E\ :\ pq=0\in\mathrm{End}(V)\}$ (the zero level of the moment map). Prove that $(p,q)\in M^{\theta-ss}$ iff

a) p is surjective, provided $\theta=\det^{-1}$.

b) q is injective, provided $\theta = \det$.

3. Prove that a) $\mathfrak{M}_0 := M/\!\!/G \simeq \overline{\mathbb{O}}$ is the closure of nilpotent orbit in $\mathfrak{gl}(W)$ formed by the endomorpisms x such that $x^2 = 0$ and $\operatorname{rk} x \le \dim V$.

b) $\mathfrak{M}_{-1} := M/\!\!/_{\det^{-1}}G \simeq T^*\mathrm{Gr}(\dim W - \dim V, W)$ is the cotangent bundle of the Grassmannian of subspaces in W of codimension $\dim W - \dim V$.

c) $\mathfrak{M}_1 := M/\!\!/_{\det}G \simeq T^*\mathrm{Gr}(\dim V, W)$ is the cotangent bundle of the Grassmannian of subspaces in W of codimension $\dim V$.

Thus $\mathfrak{M}_{-1}$ and $\mathfrak{M}_1$ are two *different* resolutions of $\overline{\mathbb{O}}$, provided $\dim W \ge 2\dim V$ (though they are abstractly isomorphic as varieties). This is the basic example of *flop*.

4. Let $W = \mathbb{C}^n$, $V_i = \mathbb{C}^i$ for $1 \le i \le n-1$, $G = \prod_{i=1}^{n-1} GL(V_i)$, and

$$E = \mathrm{Hom}(W, V_{n-1}) \oplus \mathrm{Hom}(V_{n-1}, W) \oplus \bigoplus_{i=1}^{n-2} (\mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i)),$$

$M = \{(p, q, A_i, B_i) \in E \ : \ A_{n-2}B_{n-2} + pq = 0,\ B_1A_1 = 0,\ A_iB_i - B_{i+1}A_{i+1} = 0,\ 1 \le i \le n-3\}$ (the zero set of the moment map $E \to \bigoplus_{i=1}^{n-1} \mathrm{End}(V_i)$). Prove that $(p, q, A_i, B_i) \in M^{\theta-ss}$ iff

a) There is no proper A, B-closed subspace of $V = \bigoplus_{i=1}^{n-1} V_i$ containing the image of p, provided $\theta = \prod_{i=1}^{n-1} \det_i^{-1}$.

b) There is no proper A, B-closed subspace of V contained in the kernel of q, provided $\theta = \prod_{i=1}^{n-1} \det_i$.

5. Prove that a) $\mathfrak{M}_0 := M/\!\!/G \simeq \mathcal{N}$ is the nilpotent cone of $\mathfrak{gl}(W)$.

b) Both $\mathfrak{M}_1 := M/\!\!/_{\det}G$ and $\mathfrak{M}_{-1} := M/\!\!/_{\det^{-1}}G$ are isomorphic to the cotangent bundle $T^*\mathcal{B}$ of the flag variety of $GL(W)$, so that the natural projection to $\mathfrak{M}_0$ is the Springer resolution.

Exercises in Invariant Theory 18.05.2021

1. Let $G = GL(V)$ act on $E = V \oplus V^* \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)$, and let $M \subset E$ be the zero level of the moment map: $M = \{(v, v^*, A, B) \ : \ v \otimes v^* + AB - BA = 0\}$. Prove that

a) M has $n+1$ irreducible components of dimension $n^2 + 2n$ (where $\dim V = n$) (in one of them a dense open subset is formed by det-stable points, in another one a dense open subset is formed by $\det^{-1}$-stable points).

b) $M \subset E$ is a complete intersection (i.e. it is equidimensional of codimension n^2 and it is cut out by n^2 equations) and the scheme M is reduced (its defining ideal is radical).

2. For $\tau \in \mathbb{C}$ consider a deformation $E \supset M_\tau := \{(v, v^*, A, B) \ : \ v \otimes v^* + AB - BA = \tau\,\mathrm{Id}\}$. Prove that for $\tau \ne 0$

a) M_τ is a connected smooth variety of dimension $n^2 + 2n$.

b) All the points of M_τ are both det- and $\det^{-1}$-semistable.

c) The action of $GL(V)$ on M_τ is free.

The quotient $M_\tau/GL(V)$ is the completed phase space of the Caloger-Moser integrable system, as constructed by D. Kazhdan, B. Kostant and S. Sternberg.

3. Let X be an affine algebraic variety equipped with an action of a reductive algebraic group G. Show that there are finitely many reductive subgroups $H_1, \ldots, H_k \subset G$ such that every closed G-orbit in X is G-equivariantly isomorphic to one of G/H_i.

4. Construct a surjective homomorphism with kernel $\{\pm 1\}$:
a) $SL_4 \twoheadrightarrow SO_6$.
b) $Sp_4 \twoheadrightarrow SO_5$.
c) $SL_2 \times SL_2 \twoheadrightarrow SO_4$.
d) $SL_2 \twoheadrightarrow SO_3$.

5. The 4-dimensional fundamental representation of Sp_4 will be denoted V_1, and the 5-dimensional fundamental representatoin of Sp_4 (factoring through SO_5) will be denoted V_2. We have $V_2 \subset \Lambda^2 V_1$, and hence the projection $p\colon V_2 \otimes V_2 \to \mathbb{C}$ arising from $\Lambda^4 V_1 = \mathbb{C}$. Also we have the projection $q\colon V_1 \otimes V_2 \to \Lambda^3 V_1$. Prove that the Plücker equations in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ for the flag variety $\mathcal{B}$ of Sp_4 are $p = q = 0$.

Exercises in Invariant Theory 25.05.2021

1. We consider $V_0 = \mathbb{C}^m$ with a nondegenerate symmetric bilinear form $(,)$, and $V_1 = \mathbb{C}^{2n}$ with a nondegenerate skew-symmetric bilinear form $\langle,\rangle$. The tensor product $V_0 \otimes V_1$ is equipped with a nondegenerate skew-symmetric bilinear form $(,)\otimes\langle,\rangle$ and with a hamiltonian action of $SO(V_0) \times Sp(V_1)$. Using the identification $V_0 \cong V_0^*$, $V_1 \cong V_1^*$ we can identify $V_0 \otimes V_1 \cong V_0^* \otimes V_1 = \mathrm{Hom}(V_0, V_1)$. Given $A \in \mathrm{Hom}(V_0, V_1)$ we have the adjoint operator $A^t \in \mathrm{Hom}(V_1, V_0)$.

Prove that a) $\mathbf{q}_0\colon V_0 \otimes V_1 \to \mathfrak{so}(V_0) \cong \mathfrak{so}(V_0)^*,\ A \mapsto A^t A$ is the moment map, where we identify $\mathfrak{so}(V_0) \cong \mathfrak{so}(V_0)^*$ via the trace form of the defining representation.

b) $\mathbf{q}_1\colon V_0 \otimes V_1 \to \mathfrak{sp}(V_1) \cong \mathfrak{sp}(V_1)^*,\ A \mapsto AA^t$ is the moment map, where we identify $\mathfrak{sp}(V_1) \cong \mathfrak{sp}(V_1)^*$ via the negative trace form of the defining representation.

2. Prove that a) the total moment map $(\mathbf{q}_0, \mathbf{q}_1)\colon V_0 \otimes V_1 \to \mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ coincides with half-self-supercommutator on the odd part of $\mathfrak{osp}(V_0|V_1)$.

b) The natural action of $\mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ on $V_0 \otimes V_1$ coincides with the (super)commutator bracket between the even and odd parts of $\mathfrak{osp}(V_0|V_1)$.

3. a) Let $m \leq 2n+1$. Prove that the moment map $\mathbf{q}_0$ induces an isomorphism of categorical quotients $(V_0 \otimes V_1)/\!\!/(SO(V_0) \times Sp(V_1)) \simeq \mathfrak{so}(V_0)^*/\!\!/SO(V_0) \cong \Sigma_0$ (Kostant slice).

b) Let $m \geq 2n+1$. Prove that the moment map $\mathbf{q}_1$ induces an isomorphism of categorical quotients $(V_0 \otimes V_1)/\!\!/(SO(V_0) \times Sp(V_1)) \simeq \mathfrak{sp}(V_1)^*/\!\!/Sp(V_1) \cong \Sigma_1$.

4. Let $\delta_1, \ldots, \delta_n$ be a basis of a dual Cartan Lie algebra $\mathfrak{t}_1^* \subset \mathfrak{sp}(V_1)^*$, so that the Weyl group W_1 acts by the permutations and sign changes of basis elements. Let $\varepsilon_1, \ldots, \varepsilon_k$ be a basis of a dual Cartan Lie algebra $\mathfrak{t}_0^* \subset \mathfrak{so}(V_0)^*$ (where $k = \lfloor \frac{m}{2} \rfloor$), so that the Weyl group W_0

acts by permutations and sign changes of (an even number of) basis elements (if m is even). We have $\Sigma_0 = \mathfrak{t}_0^*/\!\!/W_0,\ \Sigma_1 = \mathfrak{t}_1^*/\!\!/W_1$. In case $m \leq 2n+1$, the morphism $\mathfrak{t}_0^* \to \mathfrak{t}_1^*,\ \varepsilon_i \mapsto \delta_i$, gives rise to $\varpi_{01}\colon \Sigma_0 \to \Sigma_1$ (e.g. when $m = 2n,\ \varpi_{01}$ is a two-fold cover). In case $m \geq 2n+1$, the morphism $\mathfrak{t}_1^* \to \mathfrak{t}_0^*,\ \delta_i \mapsto \varepsilon_i$, gives rise to $\varpi_{10}\colon \Sigma_1 \to \Sigma_0$. Prove that the following diagrams commute:

$$\begin{tikzcd}
(V_0 \otimes V_1)/\!\!/(SO(V_0) \times Sp(V_1)) \arrow[r, "\mathbf{q}_1"] \arrow[d, "\mathbf{q}_0"', "\wr"] & \mathfrak{sp}(V_1)^*/\!\!/Sp(V_1) \arrow[d, equal] \\
\Sigma_0 \arrow[r, "\varpi_{01}"] & \Sigma_1,
\end{tikzcd} \qquad (m \leq 2n+1),$$

$$\begin{tikzcd}
(V_0 \otimes V_1)/\!\!/(SO(V_0) \times Sp(V_1)) \arrow[r, "\mathbf{q}_0"] \arrow[d, "\mathbf{q}_1"', "\wr"] & \mathfrak{so}(V_0)^*/\!\!/SO(V_0) \arrow[d, equal] \\
\Sigma_1 \arrow[r, "\varpi_{10}"] & \Sigma_0,
\end{tikzcd} \qquad (m \geq 2n+1).$$

5. Prove that the null-cone $\mathcal{N} \subset V_0 \otimes V_1$ contains an open $SO(V_0) \times Sp(V_1)$-orbit (in fact, it consists of finitely many orbits).

Exercises in Invariant Theory 01.06.2021

1. Choose an element u in the open orbit of Problem 5 of May 25th.

a) Prove that there exists a rational semisimple element $h \in \mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ such that $hu = 2u$.

b) Let $c_1 \geq c_2 \geq \ldots \geq c_r$ be the eigenvalues of h in the fiber L of the normal bundle to $\mathcal{N}$ at u. More precisely, we view L as an h-invariant vector subspace of $V_0 \otimes V_1$ such that $L \oplus T_u\mathcal{N} = V_0 \otimes V_1$ (where we view the tangent space $T_u\mathcal{N}$ also as an h-invariant subspace of $V_0 \otimes V_1$). Prove that $c_1 \leq 0$.

c) Let $d_1 \leq d_2 \leq \ldots \leq d_r$ be the degrees of the generators of $\mathbb{C}[V_0 \otimes V_1]^{SO(V_0)\times Sp(V_1)}$ as in Problem 3 of May 25th. Prove that $c_i = 2 - 2d_i,\ i = 1, \ldots, r$.

d) Prove that $\Sigma := u + L$ is a *Weierstraß section*, i.e. the composition $\Sigma \hookrightarrow V_0 \otimes V_1 \twoheadrightarrow (V_0 \otimes V_1)/\!\!/(SO(V_0) \times Sp(V_1))$ is an isomorphism, and Σ lies in the open subset $(V_0 \otimes V_1)^{\mathrm{reg}}$ of *regular* elements (the ones with stabilizer of minimal dimension; equivalently, with the orbit of maximal dimension).

2. a) Let $m < 2n+1$. Prove that

$$\mathbf{q}_1(A) \in \mathfrak{sp}(V_1)^{*\,\mathrm{reg}} \implies A \in (V_0 \otimes V_1)^{\mathrm{reg}} \implies \mathbf{q}_0(A) \in \mathfrak{so}(V_0)^{*\,\mathrm{reg}}.$$

b) Let $m \geq 2n+1$. Prove that

$$\mathbf{q}_0(A) \in \mathfrak{so}(V_0)^{*\,\mathrm{reg}} \implies A \in (V_0 \otimes V_1)^{\mathrm{reg}} \implies \mathbf{q}_1(A) \in \mathfrak{sp}(V_1)^{*\,\mathrm{reg}}.$$

3. Prove that the variety of complete self-orthogonal flags in V_0 has two connected components if m is even.

4. Let $\mathcal{B}_1$ (resp. $\mathcal{B}_0$) denote the variety of complete self-orthogonal flags in V_1 (resp. a connected component of the variety of such flags in V_0). A point in $\mathcal{B}_{0,1}$ will be denoted $F_{0,1} = \left(F_{0,1}^{(1)} \subset F_{0,1}^{(2)} \subset \ldots \subset F_{0,1}^{(\dim V_{0,1})} = V_{0,1}\right)$. For a point $(F_0, F_1) \in \mathcal{B}_0 \times \mathcal{B}_1$ we consider a subvariety $\mathcal{N}_{F_0,F_1} \subset \mathcal{N} \subset V_0 \otimes V_1$ formed by all $A \in V_0 \otimes V_1 \cong \mathrm{Hom}(V_0, V_1)$ such that

a) $AF_0^{(i)} \subset F_1^{(i-1)}$ in the case $m = 2n + 1$.

b) $AF_0^{\prime(i)} \subset F_1^{(i-1)}$ in the case $m = 2n + 2$ provided we slightly modify the self-orthogonal flag F_0 disregarding its middle component (that is uniquely determined by $F_0^{(n)} \subset F_0^{(n+2)}$ and our choice of connected component anyway), setting $F_0^{\prime(i)} = F_0^{(i)}$ for $i \leq n$, and $F_0^{\prime(i)} = F_0^{(i+1)}$ for $i > n$.

c) $AF_0^{(i)} \subset F_1^{(i)}$ for $i \leq n$, and $AF_0^{(i)} \subset F_1^{(i-1)}$ for $i > n$, in the case $m = 2n$.

Prove that $\mathcal{N}_{F_0,F_1}$ is a Lagrangian vector subspace of $V_0 \otimes V_1$ (it is the odd part of the nilpotent radical of an appropriate Borel subalgebra of $\mathfrak{osp}(V_0|V_1)$).

5. In the three cases of Problem 4 we obtain a vector bundle $\widetilde{\mathcal{N}}$ over $\mathcal{B}_0 \times \mathcal{B}_1$ (with the fiber $\mathcal{N}_{F_0,F_1}$ over $(F_0, F_1) \in \mathcal{B}_0 \times \mathcal{B}_1$) equipped with a forgetting (of the flags) morphism $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$. Prove that π is a resolution of singularities (i.e. π is birational and surjective).

Exercises in Invariant Theory 08.06.2021

1. We extend a nondegenerate quadratic form q on a vector space V to a quadratic form Λq on $\Lambda^\bullet V$ by the formula $\Lambda q(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k) = \det\left(q(v_i, w_j)\right)$. The operator of left multiplication by $u \in \Lambda^\bullet V$ will be denoted $\ell(u)$, and the adjoint operator (with respect to Λq) will be denoted $\iota(u)\colon \Lambda^\bullet V \to \Lambda^\bullet V$.

a) Prove that for $v, w \in V \subset \Lambda^\bullet V$ we have $\ell(v)\iota(w) + \iota(w)\ell(v) = q(v, w)$.

b) Set $\gamma(v) := \ell(v) + \iota(v)$. Prove that $\gamma(v)^2 = q(v, v)$. Hence a map $v \mapsto \gamma(v)$, $V \to \mathrm{End}\,\Lambda^\bullet V$, extends to a homomorphism $\gamma\colon C(q) \to \mathrm{End}\,\Lambda^\bullet V$, $v_1 \cdots v_k \mapsto (\ell(v_1) + \iota(v_1)) \circ \ldots \circ (\ell(v_k) + \iota(v_k))$.

c) Prove that a map $\psi\colon C(q) \to \Lambda^\bullet$, $x \mapsto \gamma(x)(1)$, is an isomorphism (of vector spaces, not algebras).

2. Prove that a) the intersection of the center of the Clifford algebra $C(q)$ with the even subalgebra C^+ consists of scalars $\mathbb{C}$.

b) If $x \in C^-$ enjoys the property $x \cdot v = -v \cdot x$ for all the vectors $v \in V$, then $x = 0$.

3. Let $C(m, n)$ be the real Clifford algebra of a quadratic form with m positive squares and n negative squares. Prove that

a) $C(0, 1) \simeq \mathbb{C}$.

b) $C(0, 2) \simeq \mathbb{H}$ (Hamilton quaternions).

c) $C(n, n) \simeq \mathrm{Mat}_{2^n}(\mathbb{R})$.

d) $C(n + 1, n) \simeq \mathrm{Mat}_{2^n}(\mathbb{R}) \oplus \mathrm{Mat}_{2^n}(\mathbb{R})$.

e) $C(m, n)$ is isomorphic to one or two copies of a matrix algebra over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

4. a) Let $v, w \in V$; $q(v,v) = q(w,w) \neq 0$. Prove that v can be transformed to w either by one reflection or by the composition of two reflections.

b) Prove that an arbitrary element of $O(V)$ is the product of at most $2\dim V$ reflections.

5. Prove that the center of $\mathrm{Spin}_m(\mathbb{C})$ is equal to

a) $\{\pm 1\}$, if m is odd.

b) $\{\pm 1, \pm\omega\}$, if $m = 2n \geq 4$ is even, where

$$\omega = (-2\sqrt{-1})^{-n}(v_1 \cdot v_{n+1} - v_{n+1} \cdot v_1) \dots (v_n \cdot v_{2n} - v_{2n} \cdot v_n).$$

Exercises in Invariant Theory 15.06.2021

1. Prove that a) the image of the spinor representation $\mathrm{Spin}_m \to GL(S)$ lies in $SL(S)$.

b) If $m = 2n \geq 4$, then the images of half-spinor representations $\mathrm{Spin}_m \to GL(S^\pm)$ lie in $SL(S^\pm)$.

2. We define a nondegenerate bilinear form β on the spinor space $S = \Lambda^\bullet L$: $\beta(s,t) := \mathrm{pr}(\tau(s) \wedge t)$. Here $\tau(v_1 \wedge \ldots \wedge v_r) := v_r \wedge \ldots \wedge v_1$ is an antiinvolution, and pr is the projection onto the top degree $\Lambda^{\mathrm{top}} L \simeq \mathbb{C}$. Let $m = 2n$.

a) Prove that $\beta(s,t)f = \tau(s \cdot f) \cdot t \cdot f$ for an appropriate generator $f \in \Lambda^n L'$; hence the action of $\mathrm{Spin}(q)$ on S preserves β.

b) Prove that β is symmetric (resp. skew-symmetric) if $n \equiv 0, 1 \pmod 4$ (resp. $n \equiv 2, 3 \pmod 4$).

c) If n is odd, then the restrictions $\beta|_{S^\pm} \equiv 0$.

d) If n is even, half-spinor representations give rise to homomorphisms $\mathrm{Spin}_{2n} \to SO_{2^{n-1}}$ for $n \equiv 0 \pmod 4$ and $\mathrm{Spin}_{2n} \to Sp_{2^{n-1}}$ for $n \equiv 2 \pmod 4$. In particular, Spin_8 has two more homomorphisms onto SO_8, not to mention the tautological one (this is the base of the *triality* phenomenon).

3. Recall the setup of Problem 4 of May 18. Prove that the spinor (half-spinor) representations give rise to the isomorphisms

a) $\mathrm{Spin}_2 \cong GL(S^+) = GL_1 = \mathbb{C}^*$.

b) $\mathrm{Spin}_3 \cong SL(S) = SL_2$.

c) $\mathrm{Spin}_4 \cong SL(S^+) \times SL(S^-) = SL_2 \times SL_2$.

d) $\mathrm{Spin}_5 \cong Sp(S) = Sp_4$.

e) $\mathrm{Spin}_6 \cong SL(S^+) = SL_4$.

4. Prove that if $q(v,v) = q(w,w) = -1$, $q(v,w) = 0$, then the path $t \mapsto (\cos(t)v + \sin(t)w) \cdot (\cos(t)v - \sin(t)w)$, $0 \leq t \leq \pi/2$, connects -1 and $1 \in \mathrm{Spin}(q)$.

5. We denote by C_m the Clifford algebra of the standard quadratic form q_m on $\mathbb{C}^m$. The embedding $\mathbb{C}^{2n} = L \oplus L'$ into $\mathbb{C}^{2n+1} = L \oplus L' \oplus U$ gives rise to an embedding of C_{2n} into C_{2n+1} and of Spin_{2n} into Spin_{2n+1}. Also, $\mathbb{C}^{2n+1} = L \oplus L' \oplus U$ is embedded into $\mathbb{C}^{2n+2} = L \oplus L' \oplus U_1 \oplus U_2$, where $U_1 \oplus U_2 = \mathbb{C} \oplus \mathbb{C}$ is equipped with a quadratic form $h(1_1, 1_1) = h(1_2, 1_2) = 0$, $h(1_1, 1_2) = 1$, and $U = \mathbb{C}$ is embedded into $U_1 \oplus U_2$ so that 1

goes to $(1/\sqrt{2}, 1/\sqrt{2})$. Thus C_{2n+1} is embedded into C_{2n+2}, and Spin_{2n+1} is embedded into Spin_{2n+2}. Prove that

a) The restriction of the spinor representation S of Spin_{2n+1} onto Spin_{2n} is isomorphic to the direct sum of half-spinor representations $S^+ \oplus S^-$ of the group Spin_{2n}.

b) The restriction of both half-spinor representations of Spin_{2n+2} onto Spin_{2n+1} is isomorphic to the spinor representation of Spin_{2n+1}.