Elliptic Functions

Weierstrass $\wp$-function

Recall that

1. an elliptic function $f(u)$ is holomorphic $\Longrightarrow$ constant.
2. $\nexists$ an elliptic function of order 1.

$$\Longrightarrow \text{The simplest non-trivial elliptic function has } \begin{cases} \text{one double pole,} \\ \text{or} \\ \text{two simple poles,} \end{cases}$$

in a period parallelogram; $\wp(u)$ is the former, $\mathrm{sn}(u)$ is the latter.

We have defined $\wp(u)$ as

$$\textit{the inverse function of } u(z) = \int_{\infty}^{z} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}.$$

Here we construct it as a doubly periodic funtion by a series.

§9.1 Construction of Weierstrass $\wp$-function

Notations:

- $\Omega_1, \Omega_2 \in \mathbb{C}$: linearly independent over $\mathbb{R}$.
- $\Gamma := \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2$.

Goal: Construct a "simple" elliptic function with double poles at Γ.

An elliptic function $f(u)$ with poles of order n at Γ is expanded as:

$$f(u) = \frac{c}{(u - m_1\Omega_1 - m_2\Omega_2)^n} + \cdots$$

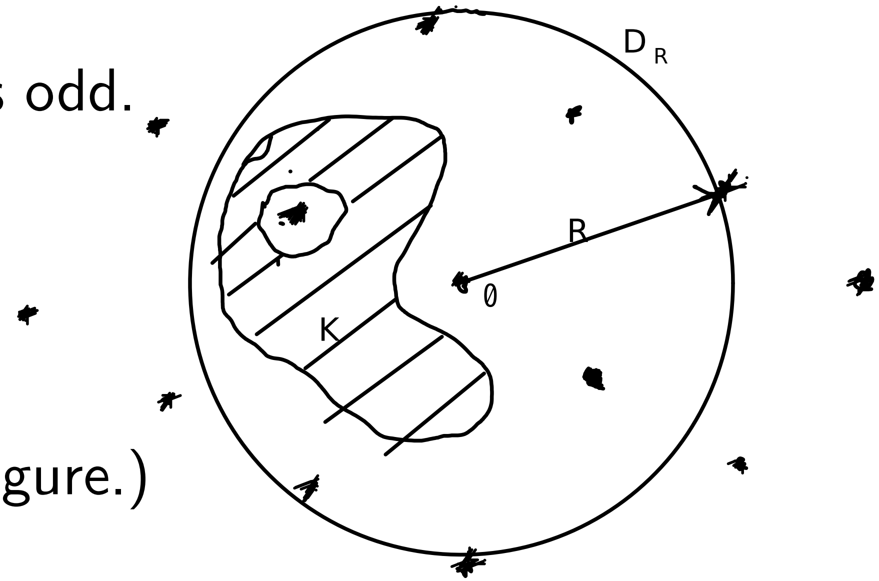
at $u = m_1\Omega_1 + m_2\Omega_2 \in \Gamma$.

$\Longrightarrow$ The simplest candidate of elliptic functions with poles of order n at Γ:

$$f_n(u) := \sum_{m_1, m_2 \in \mathbb{Z}} \frac{1}{(u - m_1\Omega_1 - m_2\Omega_2)^n}.$$

Theorem: Assume $n \geq 3$.

- The series $f_n(u)$ converges absolutely and uniformly on any compact set in $\mathbb{C} \setminus \Gamma$.
- $f_n(u)$ is an elliptic function with poles of order n at Γ.
- $f_n(u)$: even when n is even, odd when n is odd.



Proof:

$K \subset \mathbb{C} \setminus \Gamma$: compact.

$D_R := \{z \in \mathbb{C} \mid |z| \leq R\}$: a closed disk. (cf. Figure.)

R: sufficiently large so that $K \subset D_R$.

Enough to show: $f_{n,R}(u) := \displaystyle\sum_{\Omega \in \Gamma,\ \Omega \notin D_{2R}} \frac{1}{(u-\Omega)^n}$ converges absolutely and uniformly on D_R. ($f_n = f_{n,R} +$ (finite terms).)

$$\frac{1}{(u-\Omega)^n} = \frac{1}{\Omega^n}\frac{1}{(\frac{u}{\Omega}-1)^n}.$$

Lemma:

1) $\exists M > 0$ s.t. $\left|\frac{1}{(\frac{u}{\Omega}-1)^n}\right| < M$ for $u \in D_R$, $\Omega \in \Gamma \setminus D_{2R}$.

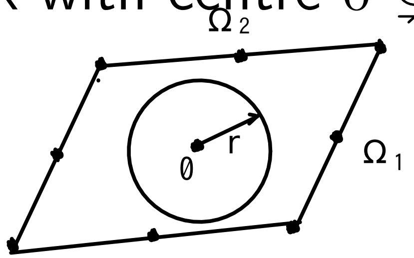
2) $\sum_{\Omega\in\Gamma,\ \Omega\neq 0} \frac{1}{|\Omega|^n}$ converges for $n \geq 3$.

Weierstrass's M-test $\Longrightarrow f_{n,R}$ converges absolutely and uniformly on D_R.

Proof of 2):

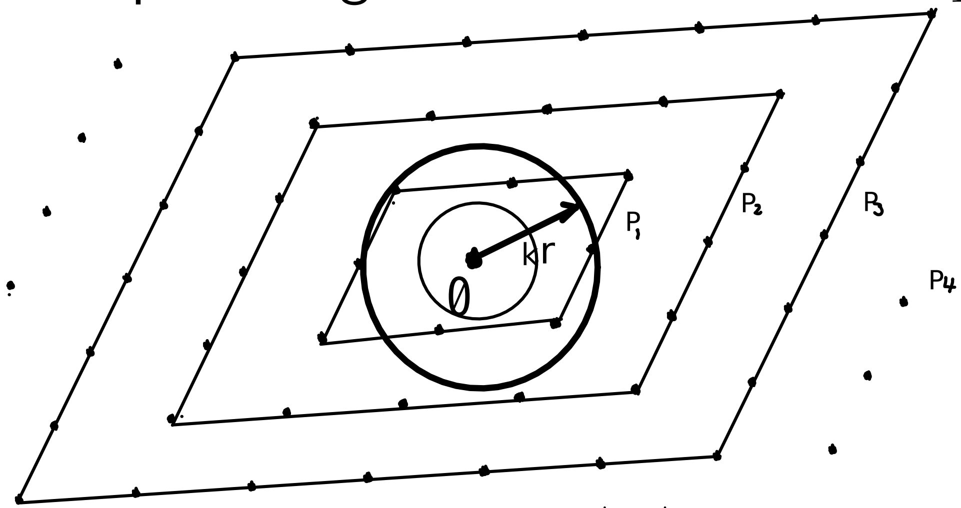
r: radius of a disk with centre $0 \subsetneq$ parallelogram with vertices $\pm\Omega_1 \pm \Omega_2$.

(Figure).



P_k: $\Gamma \cap$ (boundary of the parallelogram with vertices $\pm k\Omega_1 \pm k\Omega_2$).

(Figure)



$$\Omega \in P_k \;\Rightarrow\; kr < |\Omega|.$$

$\Longrightarrow$

$$\sum_{\Omega\in\Gamma,\ \Omega\neq 0} \frac{1}{|\Omega|^n} = \sum_{k=1}^{\infty} \sum_{\Omega\in P_k} \frac{1}{|\Omega|^n} < \sum_{k=1}^{\infty} 8k \frac{1}{k^n r^n} = \frac{8}{r^n} \sum_{k=1}^{\infty} \frac{1}{k^{n-1}},$$

which converges when $n \geq 3$. $\square$Lemma 2)

The second statement of the theorem $\Longleftarrow$ the first:

- each summand in f_n is holomorphic in $\mathbb{C} \smallsetminus \Gamma$

  $\Longrightarrow f_n$ is holomorphic in $\mathbb{C} \smallsetminus \Gamma$. (Weierstrass' theorem).

- $\dfrac{1}{(u-\Omega)^n}$ has a pole of order n at $\Omega \in \Gamma$.

The third statment:

$$f_n(-u) = \sum_{\Omega \in \Gamma} \frac{1}{(-u-\Omega)^n} = \sum_{\Omega'(=-\Omega) \in \Gamma} \frac{(-1)^n}{(u-\Omega')^n} = (-1)^n f_n(u).$$

$\square$Theorem

However, $\displaystyle\sum_{\Omega \in \Gamma,\ \Omega \neq 0} \frac{1}{|\Omega|^2}$ diverges! $\Longrightarrow$ Theorem is not true for $n = 2$.

Need "correction" to each summand.

Theorem: The series

$$\wp(u) := \frac{1}{u^2} + \sum_{\Omega\in\Gamma,\ \Omega\neq 0} \left(\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} \right)$$

- converges absolutely and uniformly on any compact set in $\mathbb{C} \setminus \Gamma$.
- gives an even elliptic function with poles of order 2 at Γ.

Namely,

$\wp(u)$ is an elliptic function of order 2: *Weierstrass's $\wp$-function*.

Proof:

We know: $f_3(u) = \displaystyle\sum_{\Omega\in\Gamma} \frac{1}{(u-\Omega)^3}$ is an elliptic function.

Idea: Integrate f_3 to get $\wp$!

Integrate $f_3(u)$ without the first term $\dfrac{1}{u^3}$ from 0:

$$\begin{aligned}
\int_0^u \left(f_3(v) - \frac{1}{v^3}\right) dv &= \int_0^u \left(\sum_{\Omega\in\Gamma\smallsetminus\{0\}} \frac{1}{(v-\Omega)^3}\right) dv \\
&= \sum_{\Omega\in\Gamma\smallsetminus\{0\}} \int_0^u \frac{1}{(v-\Omega)^3}\, dv \qquad (\Leftarrow \text{uniform convergence}) \\
&= -\frac{1}{2} \sum_{\Omega\in\Gamma\smallsetminus\{0\}} \left(\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2}\right).
\end{aligned}$$

(Absolutely and uniformly convergent on a compact set $\subset (\mathbb{C}\smallsetminus\Gamma)\cup\{0\}$.)

$\Longrightarrow$

- $\wp(u) = \dfrac{1}{u^2} - 2\displaystyle\int_0^u \left(f_3(v) - \frac{1}{v^3}\right) dv$: meromorphic with poles at Γ.
- $\wp'(u) = -2f_3(u)$.

- Evenness:

$$
\begin{aligned}
\wp(-u) &= \frac{1}{(-u)^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(-u-\Omega)^2} - \frac{1}{\Omega^2} \right) \\
&= \frac{1}{u^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u+\Omega)^2} - \frac{1}{\Omega^2} \right) \\
&= \frac{1}{u^2} + \sum_{\Omega' \in \Gamma \setminus \{0\}} \left(\frac{1}{(u-\Omega')^2} - \frac{1}{\Omega'^2} \right) = \wp(u), \qquad (\Omega' = -\Omega).
\end{aligned}
$$

- Periodicity:

$f_3(u)$: elliptic function

$\Longrightarrow \wp'(u+\Omega_1) = \wp'(u)$, $\wp'(u+\Omega_2) = \wp'(u)$.

$\Longrightarrow \exists C_1, C_2$: $\wp(u+\Omega_1) = \wp(u) + C_1$, $\wp(u+\Omega_2) = \wp(u) + C_2$.

Setting $u = -\frac{\Omega_i}{2}$, $C_i = \wp\left(\frac{\Omega_i}{2}\right) - \wp\left(-\frac{\Omega_i}{2}\right) \overset{\wp\text{: even}}{=} 0.$ $\square$

• Other properties of $\wp(u)$.

Laurent expansion at $u = 0$:

$$\wp(u) = \frac{1}{u^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} \right)$$
$$= \frac{1}{u^2} + c_0 + c_2 u^2 + \cdots + c_{2n} u^{2n} + \cdots .$$

$$c_{2n} = \frac{1}{(2n)!} \left. \frac{d^{2n}}{du^{2n}} \right|_{u=0} \left(\sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} \right) \right)$$
$$= \begin{cases} 0 & (n = 0), \\ (2n+1) \displaystyle\sum_{\Omega \in \Gamma \setminus \{0\}} \frac{1}{\Omega^{2n+2}} & (n \neq 0). \end{cases}$$

By convention: $g_2 := 20 c_2 = 60 \sum \frac{1}{\Omega^4}$, $g_3 := 28 c_4 = 140 \sum \frac{1}{\Omega^6}$.

With these notations,

$$\begin{aligned}
\wp(u) &= \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + O(u^6), \\
\wp'(u) &= -\frac{2}{u^3} + \frac{g_2}{10}u + \frac{g_3}{7}u^3 + O(u^5).
\end{aligned}$$

Hence,

$$\begin{aligned}
\wp'(u)^2 &= \quad \frac{4}{u^6} - \frac{2g_2}{5}\frac{1}{u^2} - \frac{4g_3}{7} + O(u), \\
-4\wp(u)^3 &= -\frac{4}{u^6} - \frac{3g_2}{5}\frac{1}{u^2} - \frac{3g_3}{7} + O(u), \\
g_2\,\wp(u) &= \qquad\qquad g_2\frac{1}{u^2} \qquad\qquad + O(u^2).
\end{aligned}$$

Summing up,

$$\wp'(u)^2 - 4\wp(u) + g_2\,\wp(u) = -g_3 + O(u).$$

$\wp'(u)^2 - 4\wp(u) + g_2\,\wp(u)$: elliptic function with possible poles at Γ.

$-g_3 + O(u)$: no pole at 0.

$\Longrightarrow$ $\wp'(u)^2 - 4\wp(u) + g_2\,\wp(u)$: elliptic function without poles = constant.

Namely, $\wp'(u)^2 - 4\wp(u) + g_2\,\wp(u) = -g_3$, or,

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\,\wp(u) - g_3.$$

This gives the equivalence of definitions:

$$\frac{d\wp}{du} = \sqrt{4\wp^3 - g_2\,\wp - g_3}, \text{ i.e., } du = \frac{d\wp}{\sqrt{4\wp^3 - g_2\,\wp - g_3}}.$$

Integrate from $u = 0$ ($\leftrightarrow \wp(u) = \infty$) to u ($\leftrightarrow \wp(u)$):

$$u = \int_{\infty}^{\wp(u)} \frac{dz}{\sqrt{4z^3 - g_2\,z - g_3}}.$$

$\Longrightarrow$ $\wp(u)$ is the inverse function of the elliptic integral!

In fact,

$$W : \mathbb{C}/\Gamma \ni u \mapsto (\wp(u), \wp'(u)) \in \bar{\mathcal{R}}$$

is the inverse of the Abel-Jacobi map AJ.

$\bar{\mathcal{R}}$: the elliptic curve = compactification of $\{(z, w) \mid w^2 = 4z^3 - g_2\, z - g_3\}$.

<u>Exercise</u>: Prove the bijectivity of W as follows:

(i) Show that W is *holomorphic* even at $u = 0$ as a map to $\bar{\mathcal{R}}$.

(ii) Show that $\wp'(\Omega_i/2) = 0$ for $i = 1, 2, 3$ ($\Omega_3 = \Omega_1 + \Omega_2$).

(iii) Show the bijectivity.

(Hint: $\wp(u)$ is of order 2, i.e., takes any value $\in \mathbb{P}^1$ twice on $\mathbb{C}/\Gamma$.)

<u>Exercise</u>: Prove that any elliptic function $f(u)$ with period Γ is expressed as follows:

$$f(u) = R_1(\wp(u)) + R_2(\wp(u))\, \wp'(u), \quad R_1, R_2 : \text{ rational functions.}$$

§9.2 Addition formulae of the $\wp$-function

Elliptic curve $\cong \mathbb{C}/\Gamma$ has an additive group structure:

$$u_1 \bmod \Gamma + u_2 \bmod \Gamma = u_1 + u_2 \bmod \Gamma.$$

$\Longrightarrow$ addition formulae of elliptic functions.

Theorem (Addition formula of $\wp$).

If $u_1 + u_2 + u_3 = 0$ (or $\equiv 0 \mod \Gamma$),

$$\begin{vmatrix} \wp'(u_1) & \wp(u_1) & 1 \\ \wp'(u_2) & \wp(u_2) & 1 \\ \wp'(u_3) & \wp(u_3) & 1 \end{vmatrix} = 0.$$

(Note: $\wp'(u_3) = -\wp'(u_1 + u_2)$, $\wp(u_3) = \wp(u_1 + u_2)$.)

Proof:

May assume $\wp(u_1) \neq \wp(u_2)$.

($\Longrightarrow$ analytically continued to all values afterwards.)

(a, b): a solution of

$$a\wp(u_1) + b = \wp'(u_1),$$
$$a\wp(u_2) + b = \wp'(u_2).$$

Explicit formulae (*not* used in the proof, used in the exercise):

$$a = \frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}, \qquad b = \frac{\wp(u_1)\wp'(u_2) - \wp'(u_1)\wp(u_2)}{\wp(u_1) - \wp(u_2)}.$$

$f(u) := \wp'(u) - a\wp(u) - b$: an elliptic function of the third order, because

- linear combination of elliptic functions.
- a third order pole (that of $\wp'(u)$) at $u = 0$.

$\Longrightarrow \exists$ three points, at which $f = 0$.

We know two of them: $f(u_1) = f(u_2) = 0$. Let us call the third one u_0.

By the general theorem for elliptic functions:

$$u_0 + u_1 + u_2 \equiv \text{(sum of poles)} = 0 \mod \Gamma.$$

$\Longrightarrow u_0 \equiv u_3 \mod \Gamma$, i.e., $f(u_3) = 0$.

$$f(u_1) = f(u_2) = f(u_3) = 0 \quad \Leftrightarrow \begin{pmatrix} \wp'(u_1) & \wp(u_1) & 1 \\ \wp'(u_2) & \wp(u_2) & 1 \\ \wp'(u_3) & \wp(u_3) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -a \\ -b \end{pmatrix} = 0.$$

$\begin{pmatrix} 1 \\ -a \\ -b \end{pmatrix} \neq 0 \Longrightarrow$ The matrix is degenerate, i.e., $\det = 0$. $\square$

Corollary:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4}\left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}\right)^2.$$

Proof: Exercise.

Hint: u_1, u_2 and u_3 satisfy

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \qquad \wp'(u) = a\wp(u) + b.$$

$\Longrightarrow$ $\wp(u_1)$, $\wp(u_2)$ and $\wp(u_3)$ satisfy a cubic equation.

Remark: The addition formula has a geometric interpretation.

(cf. Exercise.)