Elliptic Integrals and Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

  $$\begin{aligned}&\text{(your final mark)}\\ &= \min\left\{15 \times \frac{\text{total points you get}}{\text{max. possible points up to the problem } \mathbf{15}}, 10\right\}.\end{aligned}$$

  This means that **16** – **18** are bonus problems.
- This rule is subject to change and the latest rule applies.
- The deadline of **14** – **18**: 30 November 2021. (Send the scan or the photo to Takebe.)

The periods of elliptic functions in this sheet is Ω_1 and Ω_2. We denote the period lattice $\mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2$ by Γ. The notations are the same as those in the seminar on 23 November 2021.

14. (1 pt.) (i) Show that $\wp'\left(\frac{\Omega_i}{2}\right) = 0$ ($i = 1, 2, 3$; $\Omega_3 := \Omega_1 + \Omega_2$).

(ii) Show that $e_i := \wp\left(\frac{\Omega_i}{2}\right)$ satisfy the following relations.

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \qquad e_1e_2e_3 = \frac{g_3}{4}.$$

15. (1 pt.) Let $\bar{\mathcal{R}}$ be the elliptic curve, which is the compactification of $\{(z, w) \mid w^2 = 4z^3 - g_2\, z - g_3\}$. Prove that the map defined by

$$W : \mathbb{C}/\Gamma \ni u \mapsto (\wp(u), \wp'(u)) \in \bar{\mathcal{R}}$$

is an isomorphism of Riemann surfaces as follows. (In fact, this is the inverse of the Abel-Jacobi map AJ.)

(i) Show that W is *holomorphic* (even at $u = 0$) as a map to $\bar{\mathcal{R}}$. (Hint: In order to show that W is a holomorphic map in a neighbourhood of $u_0 \in \mathbb{C}/\Gamma$, one should use u as a local coordinate of $\mathbb{C}/\Gamma$ and choose an appropriate local coordinate of $\bar{\mathcal{R}}$ in a neighbourhood of $W(u_0)$.)

(ii) Show the bijectivity. (Hint: $\wp(u)$ is even and of order 2, i.e., takes any value $\in \mathbb{P}^1$ twice on $\mathbb{C}/\Gamma$. One also needs **14** (i) at several points.)

16. (1 pt.) Let $f(u)$ be an elliptic function.

(i) Suppose f is an even function and $\Omega \in \Gamma$. Show that, if $f(\Omega/2) = 0$ (resp. $\Omega/2$ is a pole of f), then $\Omega/2$ is a zero (resp. a pole) of even order.

(ii) Suppose f is an even function. Let $\{a_1, \dots, a_N\}$ be the set of all *distinct* zeros in the period parallelogram. Since f is an even function, $-a_i$ is also zeros of f. Therefore, for each i ($i = 1, \dots, N$) there exists i' ($i' = 1, \dots, N$) such that $a_{i'} \equiv -a_i \bmod \Gamma$. This $a_{i'}$ coincides a_i if and only if $2a_i \in \Gamma$. Hence we can renumber a_i's so that

$$\begin{aligned} &2a_i \in \Gamma, && i = N'+1, \dots, N-N', \\ &a_i \equiv -a_{N-i+1} \bmod \Gamma, && i = N-N'+1, \dots, N. \end{aligned}$$

Namely, we decompose the set $\{a_1, \dots, a_N\}$ of distinct zeros into two parts: N' pairs $(a_1, a_N), \dots, (a_{N'}, a_{N-N'+1})$, which satisfy $a_i + a_{N-i+1} \equiv 0$ and remaining zeros $a_{N'+1}, \dots, a_{N-N'+1}$, which satisfy $2a_i \in \Gamma$.

Similarly the set $\{b_1, \dots, b_M\}$ of all distinct poles in the period parallelogram can be decomposed into two parts:

$$\begin{aligned} &2b_j \in \Gamma, && j = M'+1, \dots, M-M', \\ &b_j \equiv -b_{M-j+1} \bmod \Gamma, && j = M-M'+1, \dots, M. \end{aligned}$$

We denote the order of a_i (resp. b_j) by n_i (resp. k_j) and define the integers m_i and l_j as follows:

$$m_i := \begin{cases} n_i & (2a_i \notin \Gamma), \\ n_i/2 & (2a_i \in \Gamma), \end{cases} \qquad l_j := \begin{cases} k_j & (2b_j \notin \Gamma), \\ k_j/2 & (2b_j \in \Gamma). \end{cases}$$

Show that there exists a complex number C such that

$$f(u) = C \frac{\prod_{i=1}^{N-N'} (\wp(u) - \wp(a_i))^{m_i}}{\prod_{j=1}^{M-M'} (\wp(u) - \wp(b_j))^{l_j}}.$$

(Hint: Show that the ratio of both sides is a holomorphic elliptic function and use Liouville's theorem.)

(iii) Show that an odd elliptic function $f(u)$ is a product of $\wp'(u)$ with a rational function of $\wp(u)$. Combining this result with (ii), show that an arbitrary elliptic function $f(u)$ is expressed as

$$f(u) = R_1(\wp(u)) + R_2(\wp(u))\wp'(u),$$

where R_1 and R_2 are rational functions. (Hint: To prove the last statement, show and use the fact that any elliptic function is a sum of an even elliptic function and an odd elliptic function.)

17. (1 pt.) Show the following addition formula, using the differential equation of $\wp(u)$ and the proof of the addition formula in the lecture:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4}\left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}\right)^2 .$$

(Hint: u_i's ($i = 1, 2, 3$; u_3 was defined in the seminar.) satisfy

$$\wp'(u_i)^2 = 4\wp(u_i)^3 - g_2\wp(u_i) - g_3, \qquad \wp'(u_i) = a\wp(u_i) + b.$$

Hence $\wp(u_i)$'s satisfy a cubic equation.)

18. (1 pt.) Re-interpreting the proof of the addtion formula of $\wp(u)$ in the seminar, show that one can define an abelian group structure of the elliptic curve $\bar{\mathcal{R}} := \overline{\{(z, w) \mid w^2 = 4z^3 - g_2 z - g_3\}}$, as follows:

(i) The unit element $\mathbf{O}$ is the point ∞ ($= [0 : 0 : 1] \in \mathbb{P}^2$).

(ii) Three points P_1, P_2, P_3 on $\bar{\mathcal{R}}$ satisfy $P_1 + P_2 + P_3 = \mathbf{O}$. $\iff$ There exists a line passing through P_1, P_2 and P_3.