

Elliptic Integrals and Elliptic Functions

Theta functions

§10.1 Definition of θ -function

Recall: everywhere holomorphic elliptic function = constant.

⇐ The condition “doubly periodic” is too strong.

Let us consider weaker condition of “*quasi-periodicity*”.

Remark & notations:

Up to now, we used $(\Omega_1, \Omega_2) \in \mathbb{C}^2$ as periods: $f(u + \Omega_i) = f(u)$ ($i = 1, 2$).

Renormalise the variable $u \mapsto u\Omega_1$. \implies periods = $(1, \Omega_2/\Omega_1)$.

May assume $\text{Im } \Omega_2/\Omega_1 > 0$. (If not, use $-\Omega_2/\Omega_1$ instead.)

Hereafter, periods = $(1, \tau)$, $\text{Im } \tau > 0$.

The period lattice $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$.

Let us find an entire function with (multiplicative) quasi-periodicity:

$$f(u+1) = f(u), \quad f(u+\tau) = e^{au+b} f(u).$$

- The parameter a cannot be arbitrary: Compute $f(u+1+\tau)$ in two ways:

$$f(u+1+\tau) = \begin{cases} f(u+\tau) = e^{au+b} f(u) \\ e^{a(u+1)+b} f(u+1) = e^{au+a+b} f(u). \end{cases}$$

$\implies e^a = 1$ (if f is not 0), i.e., $a = 2\pi ik$ ($k \in \mathbb{Z}$).

- Periodicity $f(u+1) = f(u) \implies \exists$ Fourier expansion: $f(u) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$.

Quick proof for an entire function:

$g(v) := f\left(\frac{\log v}{2\pi i}\right)$: well-defined on $\mathbb{C} \setminus \{0\}$ (\Leftarrow ambiguity of $\frac{\log v}{2\pi i} \in \mathbb{Z}$).

Laurent expansion $g(v) = \sum a_n v^n \implies f(u) = \sum a_n e^{2\pi i n u}$.

- Quasi-periodicity $f(u + \tau) = e^{2\pi iku+b} f(u) \Rightarrow$ recursion relation for a_n .

$$f(u + \tau) = \sum a_n e^{2\pi in\tau} e^{2\pi inu},$$

$$e^{2\pi iku+b} f(u) = \sum a_n e^{2\pi iku+b} e^{2\pi inu} = \sum e^b a_n e^{2\pi i(n+k)u}$$

$$\implies a_n = e^{-2\pi in\tau+b} a_{n-k}.$$

Exercise:

Show that if $k = 0$, then $f(u) = \alpha e^{2\pi inu}$ ($\exists \alpha \in \mathbb{C}, n \in \mathbb{Z}$).

- Case $k > 0$:

Fix $n = km + n_0$ ($0 \leq n_0 < k$).

$$a_n = e^{-2\pi in\tau+b} a_{n-k} = e^{-2\pi in\tau+b} e^{-2\pi i(n-k)\tau+b} a_{n-2k} = \dots$$

$$= e^{-2\pi i(n+(n-k)+\dots+(n_0+k))+mb} a_{n_0} = e^{-\pi im(m+1)k-2\pi imn_0+mb} a_{n_0}.$$

Recall: $g(v) := f\left(\frac{\log v}{2\pi i}\right) = \sum a_n v^n$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

$$a_n = \frac{1}{2\pi i} \oint_{|v|=R} \frac{g(v)}{v^{n+1}} dv \implies |a_n| \leq \frac{M}{R^n} \xrightarrow{n \rightarrow \infty} 0 \quad (M := \max_{|v|=R} |g(v)|).$$

On the other hand,

$$|a_{km+n_0}| = |a_{n_0} e^{-2\pi i m n_0 + mb}| e^{\pi m(m+1)k \operatorname{Im} \tau} \sim (\text{const.}) \times q^{m^2} \quad (q = e^{\pi k \operatorname{Im} \tau} > 1).$$

This diverges when $m \rightarrow \infty$: Contradiction! $\implies \nexists f(u)$.

• Case $k = -1$:

$$a_n = a_0 e^{\pi i n(n-1)\tau - nb}, \text{ i.e., } f(u) = a_0 \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n \left(u - \frac{b}{2\pi i} - \frac{\tau}{2}\right)}.$$

Definition: $\theta(u, \tau) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n u}$: θ -function.

$$\implies f(u) = a_0 \theta\left(u - \frac{b}{2\pi i} - \frac{\tau}{2}, \tau\right).$$

Lemma (Convergence of θ):

The series $\sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n u}$ converges absolutely and uniformly on

$$\{(u, \tau) \mid |\operatorname{Im} u| \leq C, \operatorname{Im} \tau \geq \varepsilon\} \quad (\forall C > 0, \varepsilon > 0).$$

Proof:

$$\left| e^{\pi i n^2 \tau + 2\pi i n u} \right| = e^{-\pi n^2 \operatorname{Im} \tau - 2\pi n \operatorname{Im} u} \leq e^{-\pi n^2 \varepsilon} e^{2\pi |n| C}.$$

$e^{-\pi n^2 \varepsilon} \searrow 0$ much faster than $e^{2\pi |n| C} \nearrow +\infty$.

$\implies \sum e^{-\pi n^2 \varepsilon} e^{2\pi |n| C}$ converges.

$\implies \sum e^{\pi i n^2 \tau + 2\pi i n u}$ converges absolutely and uniformly. □

By Weierstrass's theorem, the θ -function $\theta(u, \tau)$ is

- entire in u ,
- holomorphic in τ on $\mathbb{H} := \{\tau \mid \operatorname{Im} \tau > 0\}$.

$$\theta(u + 1, \tau) = \theta(u, \tau), \quad \theta(u + \tau, \tau) = e^{-\pi i \tau - 2\pi i u} \theta(u, \tau).$$

In general,

$$\theta(u + m + n\tau, \tau) = e^{-\pi i n^2 \tau - 2\pi i n u} \theta(u, \tau) \quad (m, n \in \mathbb{Z}).$$

Exercise:

Show that for $k \in \mathbb{Z}_{>0}$ the space of entire functions satisfying

$$f(u + 1) = f(u), \quad f(u + \tau) = e^{-2\pi i k u + b} f(u)$$

is of dimension k . Construct a basis of this space, using θ -functions.

We need variants of the θ -function: θ -functions with characteristics.

$a, b \in \mathbb{R}$: characteristics (Usually $a, b \in \mathbb{Q}$, most often $a, b \in \{0, \frac{1}{2}\}$).

$$\theta_{a,b}(u) = \theta_{a,b}(u, \tau) := \sum_{n \in \mathbb{Z}} e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(u+b)}.$$

Easily checked:

- $\theta_{a,b}(u, \tau)$: entire in u , holomorphic in τ on \mathbb{H} .
- $\theta_{00}(u) = \theta(u)$.
- $\theta_{a,b+b'}(u) = \theta_{a,b}(u + b')$.
- $\theta_{a+a',b}(u) = e^{\pi i a'^2 \tau + 2\pi i a'(u+b)} \theta_{a,b}(u + a'\tau)$.
- $\theta_{a+p,b+q}(u) = e^{2\pi i a q} \theta_{a,b}(u)$ for $p, q \in \mathbb{Z}$.

Hereafter only $\theta_{a,b}(u)$ with $a, b \in \{0, \frac{1}{2}\}$ are used.

\implies Shorthand notations: $\theta_{\varepsilon_1 \varepsilon_2}(u, \tau) := \theta_{\varepsilon_1/2, \varepsilon_2/2}(u, \tau)$ ($\varepsilon_1, \varepsilon_2 \in \{0, 1\}$).

Remark:

This is Mumford's notations in "Tata Lectures on Theta".

Correspondence with more common notations:

$$\theta_1(u) = -\theta_{11}(u), \theta_2(u) = \theta_{10}(u), \theta_3(u) = \theta_{00}(u), \theta_4(u) = \theta_{01}(u).$$

Relations with $\theta(u, \tau)$:

$$\theta_{00}(u) = \theta(u),$$

$$\theta_{01}(u) = \theta\left(u + \frac{1}{2}\right),$$

$$\theta_{10}(u) = e^{\pi i \tau / 4 + \pi i u} \theta\left(u + \frac{\tau}{2}\right),$$

$$\theta_{11}(u) = e^{\pi i \tau / 4 + \pi i (u + 1/2)} \theta\left(u + \frac{1 + \tau}{2}\right).$$

§10.2 Properties of θ -functions

- Quasi-periodicity and parity.

For $k, l \in \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ (i.e., “ $1 + 1 = 0$ ”), the following are easily checked:

- $\theta_{kl}(u + \frac{1}{2}) = (-1)^{kl} \theta_{k, l+1}(u)$.
- $\theta_{kl}(u + \frac{\tau}{2}) = (-i)^l e^{-\pi i \tau / 4 - \pi i u} \theta_{k+1, l}(u)$.
- $\theta_{kl}(u)$: even if $(k, l) \neq (1, 1)$, odd if $(k, l) = (1, 1)$.

- Zeros.

Lemma: $\theta_{kl}(u)$ has only one zero in each period parallelogram.

Proof:

$$\theta_{kl}(u) = (\text{non-zero function}) \times \theta(u + \text{shift}).$$

\implies sufficient to show the lemma for $\theta(u)$.

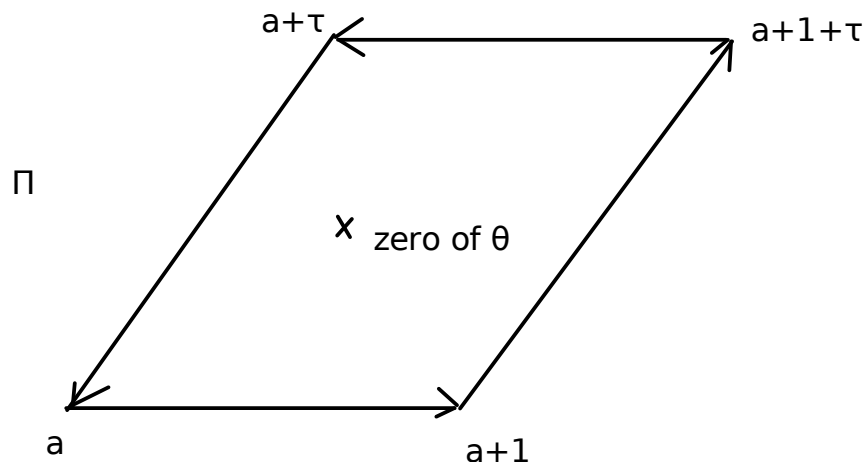
Differentiate the transformation rule:

$$\log \theta(u + 1) = \log \theta(u), \quad \log \theta(u + \tau) = \log \theta(u) - \pi i \tau - 2\pi i u.$$

$$\implies \frac{d}{du} \log \theta(u + 1) = \frac{d}{du} \log \theta(u), \quad \frac{d}{du} \log \theta(u + \tau) = \frac{d}{du} \log \theta(u) - 2\pi i.$$

Recall the argument principle: $\Pi =$ a period parallelogram (cf. Figure),

$$\# \text{ of zeros of } \theta(u) \text{ in } \Pi = \frac{1}{2\pi i} \oint_{\partial\Pi} \frac{d}{du} \log \theta(u) du.$$



$$\begin{aligned}
\oint_{\partial\Pi} \frac{d}{du} \log \theta(u) du &= \int_a^{a+1} + \int_{a+1}^{a+1+\tau} + \int_{a+1+\tau}^{a+\tau} + \int_{a+\tau}^a \\
&= \int_a^{a+1} \left(\frac{d}{du} \log \theta(u) - \frac{d}{du} \log \theta(u + \tau) \right) du \\
&\quad + \int_{a+\tau}^a \left(\frac{d}{du} \log \theta(u) - \frac{d}{du} \log \theta(u + 1) \right) du \\
&= \int_a^{a+1} (2\pi i) du = 2\pi i.
\end{aligned}$$

\implies (# of zeros of $\theta(u)$ in Π) = 1. □

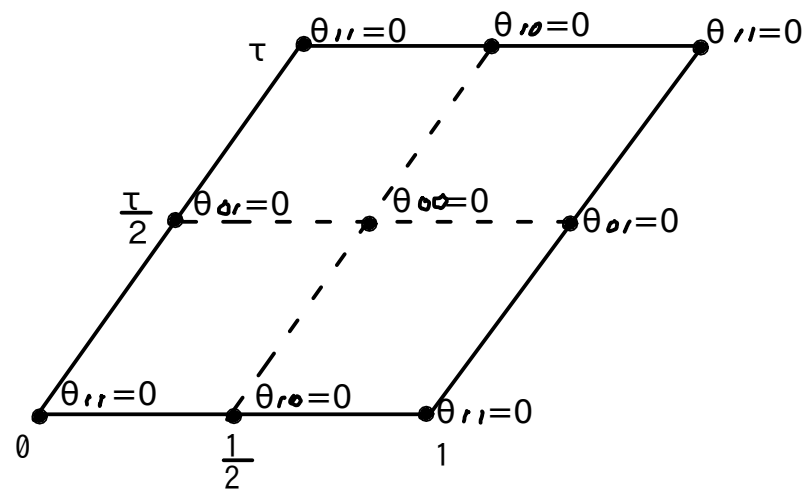
$\theta_{11}(u)$: odd $\implies \theta_{11}(0) = 0$. $\xrightarrow{\text{Lemma}} \{\text{zeros of } \theta_{11}(u)\} = \Gamma$.

Because $\theta_{10}(u) = -\theta_{11}(u + \frac{1}{2})$, $\theta_{01}(u) = (\text{non-zero}) \times \theta_{11}(u + \frac{\tau}{2})$,
 $\theta_{00}(u) = \theta_{01}(u + \frac{1}{2})$,

$$\theta_{00}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{1 + \tau}{2}, \quad \theta_{01}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{\tau}{2},$$

$$\theta_{10}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{1}{2}, \quad \theta_{11}(u) = 0 \Leftrightarrow u \in \Gamma.$$

(Figure: zeros of θ 's.)



- Jacobi's θ -relations = Analogue of addition formulae.

Theorem

Let $A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ and $\vec{y} = A\vec{x}$, where $\vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $\vec{y} := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.

Then,

$$(J0) \prod_{j=1}^4 \theta_{00}(x_j) + \prod_{j=1}^4 \theta_{01}(x_j) + \prod_{j=1}^4 \theta_{10}(x_j) + \prod_{j=1}^4 \theta_{11}(x_j) = 2 \prod_{j=1}^4 \theta_{00}(y_j).$$

Remark on the name:

- In [Mumford: Tata Lectures on Theta I]: “Riemann’s relation”.
- In [Whittaker and Watson: A Course in Modern Analysis] Jacobi’s work is cited.

Proof:

Notations: $(\vec{a}, \vec{b}) = a_1 b_1 + \cdots + a_4 b_4$ for $\vec{a}, \vec{b} \in \mathbb{C}^4$, $\prod = \prod_{j=1}^4$, $\sum = \sum_{\vec{m} \in \mathbb{Z}^4}$.

By the definition of θ 's,

$$\prod \theta_{00}(x_j) = \sum \exp(\pi i \tau(\vec{m}, \vec{m}) + 2\pi i(\vec{m}, \vec{x})),$$

$$\prod \theta_{01}(x_j) = \sum \exp(\pi i \tau(\vec{m}, \vec{m}) + 2\pi i(\vec{m}, \vec{x}) + \pi i(m_1 + \cdots + m_4)),$$

$$\prod \theta_{10}(x_j) = \sum \exp(\pi i \tau(\vec{m}', \vec{m}') + 2\pi i(\vec{m}', \vec{x})),$$

$$\prod \theta_{11}(x_j) = \sum \exp(\pi i \tau(\vec{m}', \vec{m}') + 2\pi i(\vec{m}', \vec{x}) + \pi i(m'_1 + \cdots + m'_4)),$$

where $\vec{m}' = (m_i + \frac{1}{2})_{i=1, \dots, 4}$.

When they are summed up,

- $m_1 + \cdots + m_4$ or $m'_1 + \cdots + m'_4$: odd \Rightarrow The summands cancel.
- $m_1 + \cdots + m_4$ or $m'_1 + \cdots + m'_4$: even \Rightarrow The summands are doubled.

\implies The LHS of (J0) $= 2 \sum' \exp(\pi i \tau(\vec{m}, \vec{m}) + 2\pi i(\vec{m}, \vec{x}))$,

where $\sum' =$ the sum over $\vec{m} \in \frac{1}{2}\mathbb{Z}^4$, satisfying either (i) or (ii):

(i) $\forall m_i \in \mathbb{Z}$ and $m_1 + \dots + m_4 \in 2\mathbb{Z}$ ($\Leftarrow \theta_{00}, \theta_{01}$).

(ii) $\forall m_i \in \frac{1}{2} + \mathbb{Z}$ and $m_1 + \dots + m_4 \in 2\mathbb{Z}$ ($\Leftarrow \theta_{10}, \theta_{11}$).

A : orthogonal, i.e., ${}^t A A = \text{Id}_4$. \implies For $\vec{n} := A\vec{m}$, $\vec{y} = A\vec{x}$,

- $(\vec{m}, \vec{m}) = (\vec{n}, \vec{n})$, $(\vec{m}, \vec{x}) = (\vec{n}, \vec{y})$.

- \vec{m} satisfies (i) or (ii) $\iff \vec{n} \in \mathbb{Z}^4$.

$$\begin{aligned} \sum' \exp(\pi i \tau(\vec{m}, \vec{m}) + 2\pi i(\vec{m}, \vec{x})) &= \sum_{\vec{n} \in \mathbb{Z}^4} \exp(\pi i \tau(\vec{n}, \vec{n}) + 2\pi i(\vec{n}, \vec{y})) \\ &= \prod_{j=1}^4 \theta_{00}(y_j). \quad \square \end{aligned}$$

∃ More than twenty variants. We need the following.

Corollary:

$$(J1) \quad \prod \theta_{00}(x_j) - \prod \theta_{01}(x_j) - \prod \theta_{10}(x_j) + \prod \theta_{11}(x_j) = 2 \prod \theta_{11}(y_j).$$

$$(J2) \quad \prod \theta_{00}(x_j) + \prod \theta_{01}(x_j) - \prod \theta_{10}(x_j) - \prod \theta_{11}(x_j) = 2 \prod \theta_{01}(y_j).$$

(J3)

$$\begin{aligned} & \theta_{00}(x_1)\theta_{01}(x_2)\theta_{10}(x_3)\theta_{11}(x_4) + \theta_{01}(x_1)\theta_{00}(x_2)\theta_{11}(x_3)\theta_{10}(x_4) \\ & + \theta_{10}(x_1)\theta_{11}(x_2)\theta_{00}(x_3)\theta_{01}(x_4) + \theta_{11}(x_1)\theta_{10}(x_2)\theta_{01}(x_3)\theta_{00}(x_4) \\ & = 2\theta_{11}(y_1)\theta_{10}(y_2)\theta_{01}(y_3)\theta_{00}(y_4). \end{aligned}$$

Proof:

Shift arguments in (J0):

$$x_1 \mapsto x_1 + 1 + \tau \implies \begin{cases} \theta_{00}(x_1 + 1 + \tau) = e^{-\pi i \tau - 2\pi i x_1} \theta_{00}(x_1), \text{ etc.} \\ y_j \mapsto y_j + \frac{1+\tau}{2}, \end{cases} \implies (J1).$$

$$x_1 \mapsto x_1 + 1 \implies (J2).$$

$$\left. \begin{array}{l} x_1 \mapsto x_1, \quad x_2 \mapsto x_2 + \frac{1}{2}, \\ x_3 \mapsto x_3 + \frac{\tau}{2}, \quad x_4 \mapsto x_4 + \frac{1+\tau}{2} \end{array} \right\} \implies (J3)$$

□

Notations: $\theta_{kl} := \theta_{kl}(0)$. (Note: $\theta_{11} = 0$.)

Corollary: (Addition formulae; \exists Many variants.)

$$(A1) \quad \begin{aligned} \theta_{00}(x+u)\theta_{00}(x-u)\theta_{00}^2 &= \theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2 \\ &= \theta_{01}(x)^2\theta_{01}(u)^2 + \theta_{10}(x)^2\theta_{10}(u)^2. \end{aligned}$$

$$(A2) \quad \theta_{01}(x+u)\theta_{01}(x-u)\theta_{01}^2 = \theta_{01}(x)^2\theta_{01}(u)^2 - \theta_{11}(x)^2\theta_{11}(u)^2.$$

$$(A3) \quad \theta_{11}(x+u)\theta_{01}(x-u)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(u)\theta_{11}(u) + \theta_{01}(x)\theta_{11}(x)\theta_{00}(u)\theta_{10}(u).$$

Proof:

Specialisation of x_j 's:

$$x_1 = x_2 = x, \quad x_3 = x_4 = u \implies y_1 = x + u, \quad y_2 = x - u, \quad y_3 = y_4 = 0.$$

(J1) $\xRightarrow{\text{specialisation}}$

$$\begin{aligned} & \theta_{00}(x)^2\theta_{00}(u)^2 - \theta_{01}(x)^2\theta_{01}(u)^2 - \theta_{10}(x)^2\theta_{10}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2 \\ & = 2\theta_{11}(x+u)\theta_{11}(x-u)\theta_{11}^2 = 0 \end{aligned}$$

\implies second equation in (A1).

(J0) $\xRightarrow{\text{specialisation}}$

$$\begin{aligned} & \theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{01}(x)^2\theta_{01}(u)^2 + \theta_{10}(x)^2\theta_{10}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2 \\ & = 2\theta_{00}(x+u)\theta_{00}(x-u)\theta_{00}^2. \end{aligned}$$

$$\begin{cases} \text{LHS} = 2(\theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2) & \text{(second eq. in (A1))} \\ \text{RHS} = 2 \times \text{LHS of (A1)}. \end{cases}$$

\implies first equation in (A1).

Similarly, (J2) $\xRightarrow{\text{specialisation}}$ (A2), (J3) $\xRightarrow{\text{specialisation}}$ (A3). □

- Heat equation.

The Fourier series defining θ 's converge uniformly on compact sets.

\implies May differentiate termwise.

$$\frac{\partial^2}{\partial u^2} e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(u+b)} = -4\pi^2 (n+a)^2 e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(u+b)},$$

$$\frac{\partial}{\partial \tau} e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(u+b)} = \pi i (n+a)^2 e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(u+b)}.$$

$$\implies \frac{\partial}{\partial \tau} \theta_{kl}(u, \tau) = \frac{1}{4\pi i} \frac{\partial^2}{\partial u^2} \theta_{kl}(u, \tau).$$

For $t > 0$, $x \in \mathbb{R}$, this is the *heat equation*:

$$\frac{\partial}{\partial t} \theta_{kl}(x, it) = \frac{1}{4\pi} \frac{\partial^2}{\partial x^2} \theta_{kl}(x, it).$$

- Jacobi's derivative formula.

Notations: $\theta_{kl} = \theta_{kl}(0, \tau)$ as before, $\theta'_{11} := \left. \frac{\partial}{\partial u} \right|_{u=0} \theta_{11}(u, \tau)$.

Theorem:

$$\theta'_{11} = -\pi \theta_{00} \theta_{01} \theta_{10}.$$

Proof:

(J3) with $x_1 = x$, $x_2 = x_3 = x_4 = 0$:

$$\theta_{11}(x) \theta_{10} \theta_{01} \theta_{00} = 2 \theta_{11} \left(\frac{x}{2} \right) \theta_{10} \left(\frac{x}{2} \right) \theta_{01} \left(\frac{x}{2} \right) \theta_{00} \left(\frac{x}{2} \right).$$

Substitute the Taylor expansion:

$$\theta_{kl}(x) = \theta_{kl} + \frac{\theta''_{kl}}{2} x^2 + O(x^4) \quad ((k, l) \neq (1, 1)),$$

$$\theta_{11}(x) = \theta'_{11} x + \frac{\theta'''_{11}}{6} x^3 + O(x^5).$$

(Recall: $\theta_{kl}(x)$ $((k, l) \neq (1, 1))$: even, $\theta_{11}(x)$: odd.)

Coefficients of x^3 :

$$\frac{1}{6}\theta_{11}'''\theta_{10}\theta_{01}\theta_{00} = \frac{1}{24}\theta_{11}'''\theta_{10}\theta_{01}\theta_{00} + \frac{1}{8}\theta_{11}'(\theta_{10}''\theta_{01}\theta_{00} + \theta_{10}\theta_{01}''\theta_{00} + \theta_{10}\theta_{01}\theta_{00}'').$$

$$\implies \frac{\theta_{11}'''}{\theta_{11}'} - \frac{\theta_{00}''}{\theta_{00}} - \frac{\theta_{01}''}{\theta_{01}} - \frac{\theta_{10}''}{\theta_{10}} = 0.$$

The heat equation $\implies \theta_{kl}'' = 4\pi i \frac{\partial}{\partial \tau} \theta_{kl}$ ($(k, l) \neq (1, 1)$), $\theta_{11}''' = 4\pi i \frac{\partial}{\partial \tau} \theta_{11}'$.

$$\implies 0 = \frac{\frac{\partial}{\partial \tau} \theta_{11}'}{\theta_{11}'} - \frac{\frac{\partial}{\partial \tau} \theta_{00}}{\theta_{00}} - \frac{\frac{\partial}{\partial \tau} \theta_{01}}{\theta_{01}} - \frac{\frac{\partial}{\partial \tau} \theta_{10}}{\theta_{10}} = \frac{\partial}{\partial \tau} \log \frac{\theta_{11}'}{\theta_{00} \theta_{01} \theta_{10}}.$$

$$\implies \frac{\theta_{11}'}{\theta_{00} \theta_{01} \theta_{10}} = \text{constant in } \tau.$$

The constant = the value at $\tau \rightarrow \infty$, or $q = e^{\pi i \tau} \rightarrow 0$.

Expand θ_{kl} 's and θ'_{11} in q .

$$\begin{aligned}\theta_{00} &= \sum e^{\pi i n^2 \tau} &&= 1 + O(q), \\ \theta_{01} &= \sum e^{\pi i n^2 \tau + \pi i n} &&= 1 + O(q), \\ \theta_{10} &= \sum e^{\pi i (n + \frac{1}{2})^2 \tau} &&= 2q^{1/4} + O(q), \\ \theta'_{11} &= \sum \pi i (2n + 1) e^{\pi i (n + \frac{1}{2})^2 \tau + \pi i (n + \frac{1}{2})} &&= -2\pi q^{1/4} + O(q).\end{aligned}$$

$$\frac{\theta'_{11}}{\theta_{00} \theta_{01} \theta_{10}} = \lim_{q \rightarrow 0} \frac{-2\pi q^{1/4} + O(q)}{2q^{1/4} + O(q)} = -\pi.$$

□

Remark:

θ -functions appear in

- algebraic geometry,
- number theory (especially $\theta_{ab}(0)$),
- representation theory (as characters of ∞ -dim. representations),
- mathematical physics,
- etc.

Exercise:

$$a_i, b_i, c \in \mathbb{C} \ (i = 1, \dots, N), \ \sum a_i = \sum b_i$$
$$\implies f(u) = c \frac{\theta_{11}(u - a_1) \cdots \theta_{11}(u - a_N)}{\theta_{11}(u - b_1) \cdots \theta_{11}(u - b_N)}: \text{ an elliptic function.}$$

Any elliptic function with periods 1 and τ has this form.