Seminar works 1–3: holomorphic functions, Cauchy formula, Taylor series, Hartogs theorem

February 2, 2022

Problem 1. Find the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of the following $\mathbb{R}$- linear operators $L : \mathbb{C}^2 \to \mathbb{C}$, here $z = (z_1, z_2)$, $z_k = x_k + iy_k$:

a) $L(z) = x_1 + y_1$;
b) $L(z) = x_1 + y_2$;
c) $L(z) = x_1 + 2iy_2$;
d) Homework: $(1 + i)x_1 + iy_1 + 2x_2 + 3y_2$.

Problem 2. Are the following functions of two variables $f(z) = f(z_1, z_2)$ holomorphic at the origin?

a) $f(z) = x_1 + iy_2$;
b) $f(z) = x_1^2 + 2ix_1y_1 + y_1^2$;
c) $f(z) = x_1^2 + 2ix_1y_1 - y_1^2$;
d) $f(z) = \frac{z_1+z_2}{1+z_1}$;
e) Homework $f(z) = \frac{z_1^2+z_2^3}{z_1^2+z_2^2}$;
f) Homework $f(z) = \frac{z_1^4+z_2^4}{z_1^2+z_2^2}$.

Problem 3. Find which ones of the above functions are

a) continuous in a neighborhood of zero;
b) separately holomorphic (that is, holomorphic in each individual variable z_k) in a neighborhood of the origin.

Problem 4. Calculate the following integrals:

a) $\oint_{\{|z|=1\}} \frac{\sin \zeta + 1}{\zeta} d\zeta$, $z \in \mathbb{C}$;
b) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2}{\zeta_1-\frac{1}{2}} d\zeta_1 d\zeta_2$;
c) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2+1}{\zeta_1(\zeta_2-\frac{1}{2})}$;
d) Homework $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=\frac{1}{3}\}} \frac{\cos \zeta_1+\zeta_2}{\zeta_1(\zeta_2-\frac{1}{2})}$.

Problem 5. Write the Taylor series for the following functions at the origin. Find their convergence domains and all the values of multiradii of convergence polydisks.

a) $f(z) = \frac{1}{1-z_1z_2^2}$;
b) $f(z) = \frac{1}{1-z_1-z_2^2}$;
c) $f(z) = \frac{1}{(1-z_1)(1+z_2)}$;
d) Homework $f(z) = \frac{1}{(1-(z_1+z_2)^2)(1-z_2)}$;
e) Homework $f(z) = \sin(z_1 + z_2^2)$.