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# 1. Other definitions of Milnor number of isolated singularity

1.1. **Definition 2. Index of gradient map.** $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ analytic, has isolated singularity at 0. Let $B_\varepsilon$ be a ball in $\mathbb{C}^n$ centered at 0, containing no other critical points. The gradient vector field

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

(normed) defines a continuous map $S_\varepsilon^{2n-1} \to S_1^{2n-1}$. The index of this map is equal to Milnor number.

Example: for Morse singularity, gradient map is a diffeomorphism, so index is $= 1$.

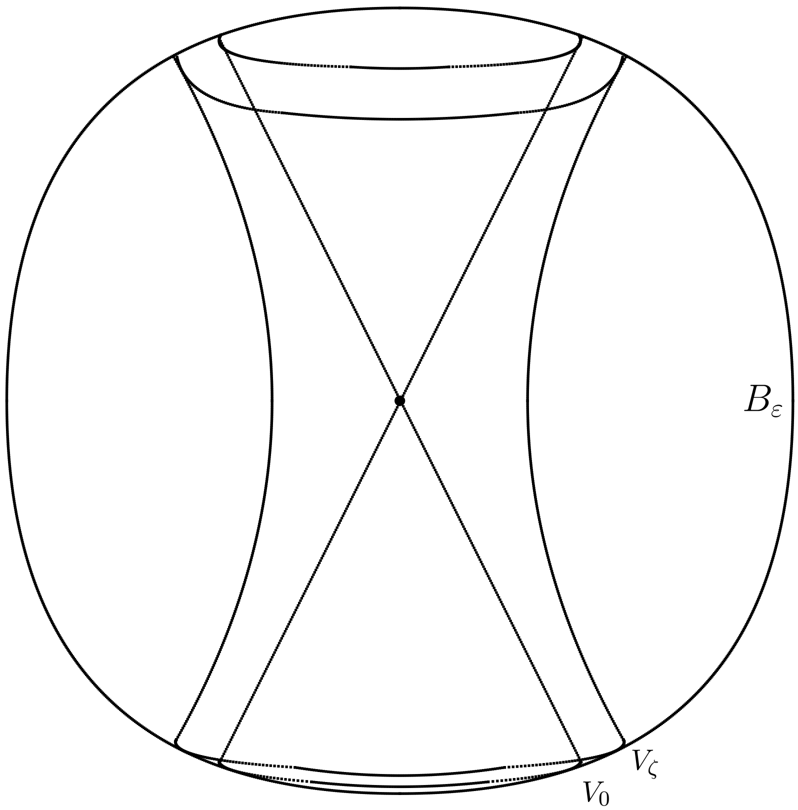
1.2. **Definition 3. Number of critical points of a small Morsification.** Let $\tilde{f}$ be a very small perturbation of $f$, so that any function of the segment $[f, \tilde{f}]$ (i.e. of form $tf + (1-t)\tilde{f}$, $t \in [0,1]$) does not have critical points at $S_\varepsilon^{2n-1}$. In particular, the gradient map of $\tilde{f}$ on $S_\varepsilon^{2n-1}$ is homotopic to that of $f$. *Milnor number* is the number of (Morse) critical points of $\tilde{f}$ inside $B_\varepsilon$.

Example: Morse singularity $f$ is already is a Morsification with a single critical point.

Equivalence of these two definitions: standard theorem on index of a vector field.

1.3. **Definition 4. Homology group of Milnor fiber.**

**Lemma 1.** *Variety $f^{-1}(0)$ is transversal to $S_\varepsilon^{2n-1}$ for any sufficiently small $\varepsilon > 0$.*




*Proof.* Suppose the converse: for any sufficiently small $\varepsilon$ there is a point $a(\varepsilon) \in S_\varepsilon^{2n-1}$ such that $f(a) = 0$ and $f^{-1}(0)$ is tangent to $S_\varepsilon^{2n-1}$ at it. Then use

**Lemma 2** (on selection of curves)**.** *If a real semi-algebraic set $A \subset \mathbb{R}^N \setminus 0$ approaches the point* $0$ *then there is a germ of analytic non-constant curve* $\varphi : [0, \tau) \to$

$(\mathbb{R}^N, 0)$ *such that* $\varphi(t) \in A$ *for any* $t \in (0, \tau)$. $\square$

Apply this lemma to the set of points at which $f^{-1}(0)$ is tangent to corresponding spheres $S^{2n-1}_{\varepsilon'}$. Then $f$ cannot be constant along this curve. $\square$

Let us fix $\varepsilon$ such that $f^{-1}(0)$ is transversal to $S^{2n-1}_{\varepsilon'}$ for all $\varepsilon' \in (0, \varepsilon]$.

Then for any $\zeta \in \mathbb{C}^1$ very close to 0, the variety $f^{-1}(\zeta)$ is also transversal to $S^{2n-1}_{\varepsilon}$ (but generally not for all $\varepsilon' < \varepsilon$). (I. e., there is $\delta > 0$ such that it is true for any $\zeta$ from the $\delta$-disc $D_\delta \subset \mathbb{C}^1$ centered at 0).

**Theorem 1** (Milnor)**.** *Manifold* $V_\zeta = f^{-1}(\zeta) \cap B_\varepsilon$, $\zeta \in D_\delta \setminus \{0\}$, *is homotopy equivalent to the wedge of finitely many spheres of dimension* $n-1$*.*

In particular, $\tilde{H}_{n-1}(V_\zeta) \simeq \mathbb{Z}^\mu$ for certain integer $\mu$, and $\tilde{H}_j(V_\zeta) = 0$ for $j \neq n-1$.

This number $\mu$ is again equal to Milnor number of $f$.

Example: series $A_\mu$ in one variable. $f^{-1}(0) \cap B_\varepsilon$ consists of $\mu+1$ points, so $\tilde{H}_{n-1} = \mathbb{Z}^\mu$.

1.4. **Example of Morse singularity.** Milnor fiber $f^{-1}(\zeta)$ of Morse singularity $f = z_1^2 + \cdots + z_n^2$, let be $\zeta = \delta > 0$. $f^{-1}(\delta) \cap \mathbb{R}^n$ is the sphere $S^{n-1}$ of radius $\sqrt{\delta}$ in $\mathbb{R}^n$. By dilation we can assume $\delta = 1$. Consider projection of $f^{-1}(1)$ to $\mathbb{R}^n$. Pre-image of $0 \in \mathbb{R}^n$ is empty: indeed, $f$ is negative on the imaginary plane $i\mathbb{R}^n$. The composition of this projection with polar projection $\mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is a fiber bundle (since our function is invariant under rotations of $\mathbb{R}^n$). Its fiber over the point $(1, 0, \ldots, 0) \in S^{n-1}$ consists of points

$(z_1, \dots, z_n)$, $z_k = x_k + iy_k$, such that $x_1 > 0$, $x_2 = \dots = x_n = 0$, $(x_1 + iy_1)^2 - y_2^2 - \dots - y_n^2 = 1$, in particular $y_1 = 0$. This is a $(n-1)$-dimensional manifold, its projection to $i\mathbb{R}^n$ along $\mathbb{R}^n$ is a diffeomorphism onto the hyperplane $\{y_1 = 0\} \subset i\mathbb{R}^n$. So entire $f^{-1}(1)$ is homotopy equivalent to $S^{n-1}$ (and diffeomorphic to $TS^{n-1}$ by multiplication of fibers by $i$).

Remembering on the small ball $B_\varepsilon$ and correspondingly small $\delta$, $f^{-1}(\delta) \cap B_\varepsilon$ is diffeomorphic to $T_1 S^{n-1}$, the space of tangent vectors of length $\leq 1$ of $S^{n-1}$.

The generator of $H_{n-1}(f^{-1}(\delta))$ defined by our small sphere is called the *vanishing cycle.*

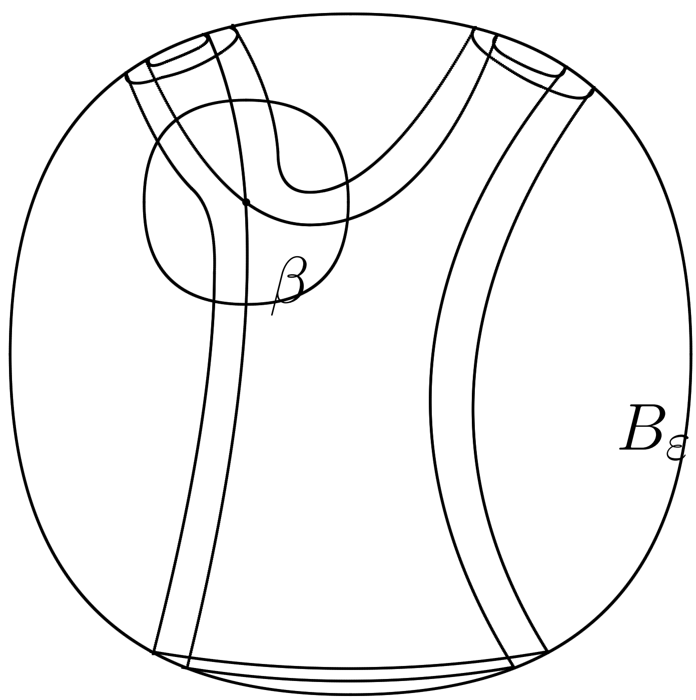
1.5. **Proof of equivalence** $3 \Leftrightarrow 4$**.** Let $\varepsilon$ be as in Lemma 1 above.

1. Let again $\tilde{f}$ be a strict Morsification of $f$ very close to $f$, then all varieties $\tilde{f}^{-1}(\eta)$ are transversal to $S_\varepsilon^{2n-1}$; if $\eta$ is not equal to either of critical values of $\tilde{f}$ then the variety $\tilde{f}^{-1}(\eta) \cap B_\varepsilon$ is non-singular and isotopic to $f^{-1}(\zeta) \cap B_\varepsilon$, $\zeta \in D_\delta \setminus \{0\}$.

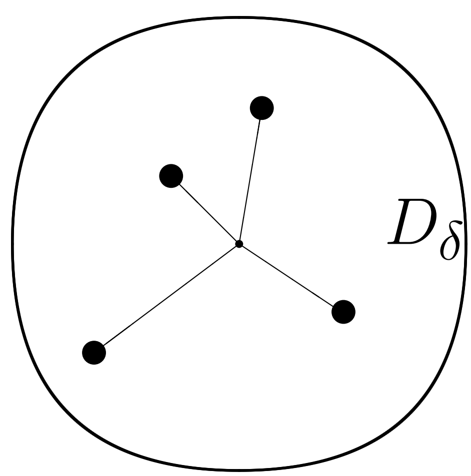
2. The singular fiber $V_0 \equiv f^{-1}(0) \cap B_\varepsilon$ is contractible. In general, if $Y$ is an algebraic subset in $\mathbb{R}^N$ or $\mathbb{C}^N$, and $a \in A$, then for any sufficiently small ball $B$ centered at $a$, the pair $(Y \cap B, Y \cap \partial B)$ is homeomorphic to the cone over $Y \cap \partial B$.

3. If $\delta$ is small enough comparing with the size of $B\varepsilon$, and $D_\delta$ is the disc or radius $\delta$ centered at 0 in $\mathbb{C}^1$, then $f^{-1}(D_\delta) \cap B_\varepsilon$ can be contracted to $V_0$ (i.e. contains it as a deformation retract), in particular also is contractible.

4. If the perturbation $\tilde{f}$ of $f$ is very small (compared with $\varepsilon$ and $\delta$), then the perturbed preimage $\tilde{V} = \tilde{f}^{-1}(D_\delta) \cap B_\varepsilon$ is homeomorphic to $f^{-1}(D_\delta) \cap B_\varepsilon$, in particular also is contractible. So, by exact sequence of the pair we need to prove that $H_n(\tilde{f}^{-1}(D_\delta) \cap B_\varepsilon, \tilde{f}^{-1}(\zeta) \cap B_\varepsilon) \sim \mathbb{Z}^\mu$ where $\mu$ is the number of critical points of $\tilde{f}$ in $B_\varepsilon$. (And we have proved it for Morse singularities.)



5. For any of $\mu$ Morse critical points of $\tilde{f}$, (call it $a_j$, with critical value $\xi_j$) consider a small ball $B_j \subset B_\varepsilon$ around it, in which the corresponding fiber $\tilde{f}^{-1}(\xi_j)$ behaves as in the Morse example (i.e. is transversal to $\partial B_j$ and to all smaller balls); let $D_j \subset D_\delta$ be a small disc around $\xi_j$ for which entire previous picture holds (i.e. all fibers $\tilde{f}^{-1}(\xi)$, $\xi \in D_j$, also are transversal to $\partial B_j$). Then by the Morse example we have $\tilde{H}_{n-1}(\tilde{f}^{-1}(\zeta) \cap B_j) \simeq \mathbb{Z}$ for any $z \in D_j \setminus \xi_j$. Also we can suppose that all discs $D_j$ do not meet one another.



Let us connect the non-critical value $\zeta$ of $\tilde{f}$ by paths with all $\mu$ discs $D_j$. Denote by $\maltese$ the union of these paths and discs $D_j$, and by $\divideontimes$ only the union of paths, without the discs. Fibers $\tilde{f}^{-1}(\xi) \cap B_\varepsilon$ form a fiber bundle over $D_\delta \setminus \{\mu$ critical values of $\tilde{f}\}$, therefore the variety $\tilde{f}^{-1}(D_\delta) \cap B_\varepsilon$ can be contracted onto $\tilde{f}^{-1}(\maltese) \cap B_\varepsilon$, in particular the latter variety also is contractible. By the same reason, the variety $\tilde{f}^{-1}(\divideontimes) \cap B_\varepsilon$ can be contracted onto $\tilde{f}^{-1}(\zeta) \cap B_\varepsilon$. So, it remains to prove that

$$H_n(\tilde{f}^{-1}(\maltese) \cap B_\varepsilon, \tilde{f}^{-1}(\divideontimes) \cap B_\varepsilon) \simeq \mathbb{Z}^\mu$$

where $\mu$ is the number of critical points $a_j$.

6. This can be proved by induction, with the Morse case as induction step. Namely, for any $k = 0, 1, \ldots, \mu$ denote by $\maltese_k$ the set $\maltese$ with first $k$ discs $D_j$ removed, in particular $\maltese = \maltese_0$ and $\divideontimes = \maltese_\mu$.

**Lemma 3.** *For any $k = 1, \ldots, \mu$, the group*

$$\tilde{H}_i(\tilde{f}^{-1}(\maltese_{k-1}) \cap B_\varepsilon, \tilde{f}^{-1}(\maltese_k) \cap B_\varepsilon)$$

*is isomorphic to $\mathbb{Z}$ if $i = n$ and is trivial for $i \neq n$.*

Then by exact sequence of this pair we get that the group $\tilde{H}_i(\tilde{f}^{-1}(\maltese_k) \cap B_\varepsilon)$ is isomorphic to $\mathbb{Z}^k$ if $i = n-1$ and is trivial for all other $i$.

*Proof of Lemma.* By excision this group is equal to $\tilde{H}_i(\tilde{f}^{-1}(D_j) \cap B_\varepsilon, \tilde{f}^{-1}(\eta_j) \cap B_\varepsilon)$, where $\eta_j$ is the common point of $D_j$ and $\divideontimes$. It is easy to see that the set $\tilde{f}^{-1}(D_j) \cap B_\varepsilon$ can be contracted onto $(\tilde{f}^{-1}(D_j) \cap B_j) \cup \tilde{f}^{-1}(\eta_j)$, and hence, again by excision, the group

$$\tilde{H}_i(\tilde{f}^{-1}(D_j) \cap B_\varepsilon, \tilde{f}^{-1}(\eta_j) \cap B_\varepsilon)$$

is isomorphic to

$$\tilde{H}_i(\tilde{f}^{-1}(D_j) \cap B_j, \tilde{f}^{-1}(\eta_j) \cap B_j),$$

which we know from the Morse case.

1.6. **Basis of vanishing cycles defined by a system of paths.** For any $j$, we get a vanishing cycle: a small sphere generating the group $\tilde{H}_{n-1}(\tilde{f}^{-1}(\eta_j) \cap B_j)$. Consider the corresponding element of group

$$\tilde{H}_{n-1}(\tilde{f}^{-1}(\eta_j) \cap B_\varepsilon).$$

Using the covering homotopy of the Milnor fibration over $D_\delta \setminus \{\text{critical values}\}$ move all these cycles to elements of

$$H_{n-1}(\tilde{f}^{-1}(\zeta) \cap B_\varepsilon)$$

along our $\mu$ paths of $\divideontimes$. Obtained $\mu$ elements $\Delta_j$ freely generate the latter group. They depend on the choice of paths (and of the orientations of small spheres).