Task 1: holomorphic functions, Cauchy formula, Taylor series. Deadline: February, 23

February 2, 2022

The **Liouville Theorem** on functions of one complex variable states that *a function holomorphic and bounded on all of* $\mathbb{C}$ *is constant.*

Prove the following extensions of the Liouville Theorem to two variables.

Problem 1. Prove that every bounded function holomorphic on $\mathbb{C}^2 \setminus K$ is constant, where

a) K is a ball;

b) K is a complex line;

d)* $K = \mathbb{R}^2 \subset \mathbb{C}^2$ is the real plane.

Problem 2. Let $V \subset \mathbb{C}^n$ be a domain, $\Gamma \subset V$ be a one-dimensional complex submanifold. Let γ be a closed path in Γ contractible along Γ (i.e., contractible as a closed path in Γ).

a) Prove that the integral along γ of each holomorphic 1-form on V vanishes.

b) Is it true that the same integrals but along *every* closed path *contractible in* V vanish?

Problem 3. Prove that the domain of convergence of any Taylor series is always *logarithmically convex:* if two points z, w are contained in the convergence domain, then for every $\alpha \in [0, 1]$ the closed polydisk $\overline{\Delta_{R(\alpha)}}$, $R_j(\alpha) := |z_j|^\alpha |w_j|^{1-\alpha}$, is also contained in the convergence domain.

Problem 4. Prove that every function holomorphic on the complement $\Delta_{(1,1)} \setminus S \subset \mathbb{C}^2$ extends holomorphically to all of $\Delta_{(1,1)}$, where

a) $S = \{\frac{1}{2} < |z_1| < 1\} \times \{0\}$;

b) $S = \mathbb{R}^2 \setminus \{|z_1|^2 + |z_2|^2 < \frac{1}{2}\}$;

c)* $S = \mathbb{R}^2$.

Hint to c). Consider the fibration of the space $\mathbb{C}^2$ by parabolas $iz_2 = z_1^2 + \varepsilon$. Try to adapt the proof of two-dimensional Hartogs Theorem (with argument on fibration by parallel lines) to this parabolic fibration.