Seminar work 3. February 15

February 15, 2022

Problem 1. Let $\Omega \subset \mathbb{C}^n$ be a connected domain. Let $F : \Omega \to \mathbb{C}^2$ be a holomorphic map whose image lies in the union of two complex lines. Prove that the latter image is contained in just one complex line.

Problem 2. Prove the following multidimensional analogue of Hartogs' erasing singularity theorem. Let $R = (R_1, \dots, R_n)$, $R_j > 0$, $1 \leq k < n$, $r = (r_1, \dots, r_k)$, $r_s < R_s$. Set $R^k = (R_1, \dots, R_k)$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. Let $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ be an open subset. Let $z = (z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$,

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}},\ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n,\ \Omega = A \cup B.$$

Then every function holomorphic on Ω extends holomorphically to the whole polydisk $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.

Problem 3. * The general Erasing Compact Singularity Theorem for a connected domain $\Omega \subset \mathbb{C}^n$ says that for every compact subset $K \Subset \Omega$ with a connected complement every function holomorphic on $\Omega \setminus K$ extends holomorphically to all of Ω. Prove it for Ω being

a) a ball $B_R = \{|z| < R\} \subset \mathbb{C}^n$;

b)* an arbitrary domain whose projection π to appropriate coordinate $n-1$-space makes it a trivial fibration by simply connected domains in $\mathbb{C}$: C^1-diffeomorphic to the direct product $D_1 \times \pi(\Omega)$.

Problem 4. Show that each holomorphic function on the complement to a complex line in

a) $\mathbb{C}^3$

b) a domain $\Omega \subset \mathbb{C}^3$

extends holomorphically to the whole domain in question ($\mathbb{C}^3$ or Ω).

Problem 5. (Homework). Prove the above extension statements for bounded holomorphic functions on complement to a complex hyperplane.