Several Complex Variables

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March 7, 2022

HSE, spring semester, 2020/2021

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1 Holomorphic functions of several complex variables. Cauchy–Riemann equations, Cauchy formula, Taylor series

Definition 1.1 Let $\Omega \subset \mathbb{C}^n$ be an open subset. Recall that a function $f : \Omega \to \mathbb{C}$ is said to be ($\mathbb{R}$-)differentiable at a point $p \in \Omega$, if it is differentiable there as a function of real variables: there exists an $\mathbb{R}$-linear mapping $df(p) : T_p\mathbb{C}^n \simeq \mathbb{R}^{2n} \to T_p\mathbb{C} \simeq \mathbb{R}^2$ such that

$$f(z) - f(p) = df(p)(z-p) + o(z-p), \text{ as } z \to p.$$

A function f is said to be $\mathbb{C}$-*differentiable at a point* p, if it is differentiable there and its differential $df(p)$ is $\mathbb{C}$-linear. A function f is said to be *holomorphic* on Ω, if it is $\mathbb{C}$-differentiable at each point $x_0 \in \Omega$. A function f is said to be *holomorphic at a point* $x_0 \in \mathbb{C}^n$, if it is $\mathbb{C}$-differentiable in some its neighborhood. A holomorphic mapping $F = (F_1, \dots, F_m) : U \to V$, $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ is defined in literally the same way: it is holomorphic, if and only if so are its components $F_1, \dots, F_m$.

Holomorphicity of a differentiable function is equivalent to Cauchy–Riemann Equations. To write them, let us first recall the following preparatory linear algebra.

Let $\mathbb{C}$ be equipped with a complex coordinate $z = x + iy$. Each $\mathbb{R}$-linear operator $L : \mathbb{C} \to \mathbb{C}$ can be written in the two following forms

$$L = \alpha x + \beta y = Az + B\bar{z}; \quad \alpha, \beta, A, B \in \mathbb{C}.$$

The expression of the coefficients A and B via α and β is obtained by the substitutions

$$x = \frac{1}{2}(z + \bar{z}), \ y = \frac{1}{2i}(z - \bar{z}) :$$

$$L = \alpha x + \beta y = \frac{\alpha}{2}(z + \bar{z}) + \frac{\beta}{2i}(z - \bar{z}) = Az + B\bar{z},$$

$$A = \frac{1}{2}(\alpha - i\beta), \ B = \frac{1}{2}(\alpha + i\bar{\beta}). \tag{1.1}$$

Let $f : U \to \mathbb{C}$ be a differentiable mapping of a domain $U \subset \mathbb{C}$. For every $p \in U$ the differential $df(p) : T_p\mathbb{C} \simeq \mathbb{C} \to T_{f(p)}\mathbb{C} \simeq \mathbb{C}$ is an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$. One has

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} \overline{dz}; \quad \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}), \tag{1.2}$$

which follows from formula (1.1) applied to the differential $L = df(p)$, taking into account that in our case

$$\alpha = \frac{\partial f}{\partial x}, \ \beta = \frac{\partial f}{\partial y}.$$

Proposition 1.2 *(Cauchy–Riemann Equations). A differentiable function $f(z_1, \dots, z_n)$ on a domain in $\mathbb{C}^n$ is holomorphic, if and only if*

$$\frac{\partial f}{\partial \bar{z}_j} \equiv 0 \text{ for every } j = 1, \dots, n. \tag{1.3}$$

The latter equation number j is equivalent to the system of equations

$$\begin{cases} \frac{\partial \operatorname{Re} f}{\partial x_j} = \frac{\partial \operatorname{Im} f}{\partial y_j} \\ \frac{\partial \operatorname{Re} f}{\partial y_j} = -\frac{\partial \operatorname{Im} f}{\partial x_j}. \end{cases} \tag{1.4}$$

Proof The tangent space $T_p\mathbb{C}^n$ is the direct sum of complex "coordinate lines" parallel to the coordinate axes. Thus, the $\mathbb{C}$-linearity of the differential $df(p)$ is equivalent to the $\mathbb{C}$-linearity of its restrictions to all the coordinate lines. The latter is equivalent to (1.3). Equivalence of equation (1.3) and system (1.4) follows from (1.2). This proves the proposition. □

Example 1.3 Holomorphicity is preserved under arithmetic combinations and compositions. In particular, polynomials and rational functions and in general, all the elementary functions (restricted to their appropriate definition domains) are holomorphic.

Remark 1.4 In the case, when $n = 1$ the above definition coincides with the classical definition of holomorphic function of one complex variable. If a function f is holomorphic in Ω, then for every complex line $L \subset \mathbb{C}^n$ the restriction $f|_{L\cap\Omega}$ is holomorphic as a function of one variable. The next Big Hartogs' Theorem implies that the converse is also true.

Theorem 1.5 *(Hartogs). A function $f(z_1, \dots, z_n)$ is holomorphic on a domain $\Omega = \Omega_1 \times \dots \times \Omega_n \subset \mathbb{C}^n$, if and only if it is* **separately holomorphic:** *for every $j = 1, \dots, n$ and every given collection of points $z_s \in \Omega_s$, $s \neq j$, the function $g(z) = f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$ is holomorphic on Ω_j.*

Remark 1.6 The nontrivial part of the theorem says that if a function is separately holomorphic, then it is holomorphic as a function of several variables. Under the additional assumption that f is differentiable, this statement follows immediately from Proposition 1.2. We will not prove Theorem

1.5 in full generality. We will prove its weaker version under continuity assumption (Osgood Lemma).

Holomorphic functions in several variables share the basic properties of holomorphic functions in one variable: existence of converging Taylor series, uniqueness of analytic extension, openness, Maximum Principle, Liouville Theorem. At the same time we will see that the following new phenomena hold for holomorphic functions in several complex variables, which are in contrast with the case of one variable:

- *no isolated singularities;*

- *erasing compact singularities:* holomorphic functions on a complement of a domain $V \subset \mathbb{C}^n$ to a compact subset $K \Subset V$ extend holomorphically to all of V.

Everywhere below for every $\delta > 0$ and $z \in \mathbb{C}$ we denote

$$D_\delta(z) = \{|w - z| < \delta\} \subset \mathbb{C};\ D_\delta = D_\delta(0).$$

The corresponding balls in $\mathbb{C}^n$ of radius δ will be denoted by $B_\delta(z)$ and B_δ respectively. For every $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the *polydisk* of multiradius r centered at z is the product of disks of radii r_j, which we will denote by

$$\Delta_r(z) = \prod_j D_{r_j}(z_j) = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n \mid |w_j - z_j| < r_j\};\ \Delta_r = \Delta_r(0).$$

For $\delta > 0$ we denote $\Delta_\delta(z) = \Delta_{(\delta,\dots,\delta)}(z)$, $\Delta_\delta = \Delta_\delta(0)$. In the case, when we would like to specify the dimension of the ambient space of the polydisk, we will write Δ^n_r, $\Delta^n_\delta(z)$ etc.

The next theorem generalizes Cauchy formula for holomorphic functions in one variable.

Theorem 1.7 *(Multidimensional Cauchy formula). Let $f : \overline{\Delta}_r \to \mathbb{C}$ be a continuous function that is holomorphic in each variable z_j, $j = 1, \dots, n$. (In particular, this holds for every function holomorphic on Δ_r and continuous on its closure). Then for every $z = (z_1, \dots, z_n) \in \Delta_r$ one has*

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1| = r_1} \cdots \oint_{|\zeta_n| = r_n} \frac{f(\zeta)}{\prod_{j=1}^n (\zeta_j - z_j)} d\zeta_n \dots d\zeta_1. \tag{1.5}$$

Remark 1.8 Let $g(\zeta)$ denote the sub-integral function in the latter right-hand side. The multiple integral in (1.5) is independent of integration order (Fubini's theorem and continuity of the function $g(\zeta)$). It is equal to the integral of the complex-valued differential n-form $g(\zeta)d\zeta_1 \wedge \cdots \wedge d\zeta_n$ on the n-torus $\mathbb{T}^n = \prod_{j=1}^n S_j^1$, $S_j^1 = \{|\zeta_j| = r_j\}$, oriented as a product of positively (i.e., counterclockwise) oriented circles. That is, an orienting basis $v_1, \dots, v_n \in T_\zeta \mathbb{T}^n$ is formed by vectors $v_j \in T_{\zeta_j} S_j^1$ oriented counterclockwise.

Proof It suffices to prove the statement of the theorem in the case, when f is holomorphic in each variable on a domain containing the closed polydisk $\overline{\Delta}_r$: the general case is reduced to it via scaling the function f to $f_\varepsilon(z) = f(\varepsilon z)$, $0 < \varepsilon < 1$ (which is holomorphic in each variable on $\overline{\Delta}_r$) and passing to the limit under the integral, as $\varepsilon \to 1$. We prove formula (1.5) by induction in n.

Induction base: for $n = 1$ this is the classical Cauchy formula for one variable.

Induction step. Let formula (1.5) be proved for the given $n = k$. Let us prove it for $n = k + 1$. For every $w = (w_1, \dots, w_k) \in \mathbb{C}^k$ set

$$f_w(t) = f(w_1, \dots, w_k, t).$$

For every fixed $z_{k+1} \in D_{r_{k+1}}$ the function $g(w_1, \dots, w_k) = f_w(z_{k+1})$ is holomorphic on $\overline{\Delta}_{(r_1,\dots,r_k)}$. Hence,

$$f(z_1, \dots, z_{k+1}) = \frac{1}{(2\pi i)^k} \oint_{|\zeta_1|=r_1} \cdots \oint_{|\zeta_k|=r_k} \frac{f_\zeta(z_{k+1})}{\prod_{j=1}^k (\zeta_j - z_j)} d\zeta_k \dots d\zeta_1, \quad (1.6)$$

by the induction hypothesis. The function $f_\zeta(t)$ being holomorphic in $t \in \overline{D}_{r_{k+1}}$ for every $\zeta = (\zeta_1, \dots, \zeta_k)$, it is expressed by Cauchy Formula

$$f_\zeta(t) = \frac{1}{2\pi i} \oint_{|\zeta_{k+1}|=r_{k+1}} \frac{f_\zeta(\zeta_{k+1})}{\zeta_{k+1} - t} d\zeta_{k+1} \text{ for every } t \in D_{r_{k+1}}.$$

Substituting the latter formula with $t = z_{k+1}$ to (1.6) yields (1.5), by continuity and Fubini Theorem. □

Lemma 1.9 *(Osgood). Every continuous function on a domain in $\mathbb{C}^n$ that is holomorphic in each individual variable is holomorphic.*

Proof It sufficed to prove the statement of the lemma for a function continuous on a closed polydisk $\overline{\Delta}_r$. Then Multidimensional Cauchy Formula (1.5) holds, and its subintegral expression is a continuous family of rational functions in $z \in \Delta_r$. Therefore, the subintegral expressions are holomorphic on Δ_r. They are uniformly bounded and continuous together with derivatives on compact subsets in Δ_r. Therefore, the integral is C^1-smooth and its partial derivatives are equal to the integrals of partial derivatives in z of the subintegral expression (here one can differentiate the integral by the above boundedness and continuity statements). It satisfies Cauchy–Riemann equations, as do the subintegral functions, and hence, is holomorphic. The lemma is proved. $\square$

Theorem 1.10 *Let a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ converge uniformly on compact subsets. Then its limit is holomorphic on Ω.*

Proof Let f_m be our converging functions. Let us restrict them to a closed polydisk $\overline{\Delta} \subset \Omega$ and write multidimensional Cauchy formula for them on the polydisk Δ. For each $z \in \Delta$ its left-hand side $f_m(z)$ is a converging sequence, and so is the Cauchy integral in the right-hand side, by uniform convergence of $f_m(\zeta_1, \dots, \zeta_n)$. Therefore, the limit function satisfies the Cauchy formula as well. For every continuous function $f(\zeta_1, \dots, \zeta_n)$ the corresponding Cauchy integral is holomorphic in $z \in \Delta$. This together with the latter statement implies holomorphicity of the limit. The theorem is proved. $\square$

Set

$$\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}.$$

Theorem 1.11 *Every function f holomorphic at $0 \in \mathbb{C}^n$ is a sum of power series converging to f uniformly on a neighborhood of 0:*

$$f(z) = \sum_{k \in \mathbb{Z}^n_{\geq 0}} c_k z^k; \; c_k \in \mathbb{C}, \; z^k = z_1^{k_1} \dots z_n^{k_n}, \; c_0 = f(0). \tag{1.7}$$

Proof Fix a $\delta > 0$ such that f is holomorphic on the closed polydisk $\overline{\Delta}_\delta$. Let us show that the right-hand side of the Cauchy formula is a sum of power series converging on Δ_δ. For every ζ_j and z_j with $|z_j| < \delta = |\zeta_j|$ one has

$$\frac{1}{\zeta_j - z_j} = \zeta_j^{-1} \frac{1}{1 - \frac{z_j}{\zeta_j}} = \sum_{l=0}^{+\infty} \zeta_j^{-l-1} z_j^l. \tag{1.8}$$

This series converges absolutely uniformly on every disk $|z_j| \leq \delta'$ with $\delta' < \delta$. Hence, the product of the latter series for all $j = 1, \dots, n$ also absolutely uniformly converges to $\frac{1}{\prod_j (\zeta_j - z_j)}$ on $\Delta_{\delta'}$. Substituting formulas (1.8) for all j to (1.5) together with permutability of integration and series summation (ensured by absolute uniform convergence of subintegral series and uniform boundedness of the function on $\partial\Delta$) yields (1.7) with

$$c_k = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1|=\delta'} \cdots \oint_{|\zeta_n|=\delta'} \frac{f(\zeta)}{\zeta_1^{-k_1-1} \dots \zeta_n^{-k_n-1}} d\zeta_1 \dots d\zeta_n. \tag{1.9}$$

Substituting $k = 0$ yields $c_0 = f(0)$, by (1.5). □

Proposition 1.12 *Each holomorphic function is C^∞-smooth. If f is holomorphic on a polydisk Δ_r and continuous on its closure, then its derivatives are given by the formulas*

$$\frac{\partial^k f(z)}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} = \frac{k_1! \dots k_n!}{(2\pi i)^n} \oint_{|\zeta_1|=r_1} \cdots \oint_{|\zeta_n|=r_n} \frac{f(\zeta_1, \dots, \zeta_n)}{\prod_{j=1}^n (\zeta_j - z_j)^{k_j+1}} d\zeta_n \dots d\zeta_1. \tag{1.10}$$

Proof In the multidimensional Cauchy formula the subintegral expression a non-vanishing rational function. It is holomorphic, thus its $\frac{\partial}{\partial \bar{z}_j}$-derivatives vanish. It is differentiable infinitely many times, and its k-th derivatives, $k = (k_1, \dots, k_n)$, are equal to $\frac{k_1! \dots k_n! f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1)^{k_1+1} \dots (\zeta_n - z_n)^{k_n+1}}$. This together with Cauchy formula and uniform boundedness of every latter derivative, as $|\zeta_j| = r_j$ and z varies on a compact subset in Δ_r, implies that the corresponding derivative of the Cauchy integral is the integral of the derivative. This proves (1.10). The C^∞-smoothness statement then follows immediately. □

2 Convergence of power series. Equivalent definition of holomorphic function

Here we study convergence of power series $\sum_k c_k z^k$ and present a higher-dimensional analogue of convergence radius theorem from the theory of functions of one complex variable.

Lemma 2.1 (Abel). *Consider a power series $\sum_{k \in \mathbb{Z}^n_{\geq 0}} c_k z^k$. Let its terms $c_k z^k$ at a given point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be uniformly bounded, set $r_j = |z_j|$, $r = (r_1, \dots, r_n)$. Let $r_j > 0$ for all j. Then the series converges uniformly on compact subsets in the polydisk Δ_r.*

In the proof of the lemma and in what follows we will use the following convention.

Convention 2.2 For every $\delta, r \in \mathbb{R}^n_{\geq 0}$ we say that $\delta < r$ ($\delta \leq r$), if $\delta_j < r_j$ (respectively, $\delta_j \leq r_j$) for every $j = 1, \dots, n$.

Proof of Lemma 2.1. Fix some $\delta = (\delta_1, \dots, \delta_n)$ with $\delta_j > 0$, $\delta < r$. It suffices to show that $\sum |c_k|\delta^k < \infty$. Indeed, set

$$\nu_j = \frac{\delta_j}{r_j} < 1, \; C = \sup_k |c_k r^k| < +\infty.$$

Then $|c_k|\delta^k \leq C\nu^k$. But

$$\sum_k \nu^k = \prod_{j=1}^n (\sum_{s=0}^{+\infty} \nu_j^s) = \frac{1}{\prod_j (1-\nu_j)} < +\infty.$$

Therefore, the series $\sum_k |c_k|\delta^k$ is majorated by a converging series $C\sum_k \nu^k$, and hence, converges. The lemma is proved. □

Definition 2.3 The *convergence domain* of a power series $\sum_{k\in\mathbb{Z}^n_{\geq 0}} c_k z^k$ is the interior of the set of points $z \in \mathbb{C}^n$ where it converges.

Consider the torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$ identified with the product of unit circles in $\mathbb{C}$. Its points will be identified with collections $t = (t_1, \dots, t_n)$, $|t_1| = \cdots = |t_n| = 1$. It acts on $\mathbb{C}^n$ by coordinatewise rotations:

$$\mathbb{T}^n : \mathbb{C}^n \to \mathbb{C}^n, \; t(z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n).$$

Corollary 2.4 *The convergence domain of a series $\sum_k c_k z^k$ is a union of polydisks centered at the origin. It is invariant under the above torus action.*

Proof Given a power series, let Ω denote its convergence domain. Given a point $z = (z_1, \dots, z_n) \in \Omega$, let us construct a polydisk $\Delta_r \subset \mathcal{D}$ containing z. For every $\lambda > 1$ close enough to 1 (dependently on z) one has $w := \lambda z \in \Omega$, by definition. Fix the above λ and w. Set $r_j = |w_j| = \lambda|z_j| > |z_j|$, $r = (r_1, \dots, r_n)$. The sequence $c_k r^k$ is uniformly bounded, by the convergence of the series $\sum_k c_k w^k$. Therefore, $\Delta_r \subset \Omega$ (Abel's Lemma 2.1) and $z \in \Delta_r$, by construction. The first statement of the corollary is proved. Its second statement follows from the first one and the invariance of each polydisk centered at 0 under the torus action. The corollary is proved. □

Proposition 2.5 *Each power series converges uniformly on compact subsets in its convergence domain Ω.*

Proof Each point $z \in \Omega$ is contained in two homothetic polydisks: a polydisk $\Delta_r \subset \Omega$ and in smaller homothetic polydisk $\Delta_{r'}$, $\overline{\Delta}_{r'} \subset \overline{\Delta}_r \subset \Delta_r$, $r' = \lambda r$, $0 < \lambda < 1$. The series converges uniformly on $\overline{\Delta}_{r'}$, since it converges at the point r and by Abel's Lemma. Every compact subset $K \Subset \Omega$ can be covered by a finite number of the above uniform convergence polydisks $\Delta_{r'}$. This implies uniform convergence on K. The proposition is proved. □

Example 2.6 The convergence domain of the series $\sum_{k\geq 0} z_1^k$ in two variables (z_1, z_2) is the cylinder $|z_1| < 1$. The convergence domain of the series $\sum z_1^{k_1} z_2^{k_2}$ is the unit bidisk $\Delta_{1,1}$. The convergence domain of the series $\sum (z_1 z_2)^k$ is the set $\{|z_1 z_2| < 1\}$.

Let us recall that the convergence radius r of a power series $\sum_k c_k z^k$ in one variable is given by the classical Cauchy-Hadamard formula $r = (\overline{\lim}_{k\to\infty} c_k^{\frac{1}{k}})^{-1}$, or equivalently,

$$\overline{\lim}_{k\to\infty} (c_k r^k)^{\frac{1}{k}} = 1.$$

The next proposition generalizes this formula to several variables. To state it, let us introduce the following notation. Consider the mapping

$$R : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}, \quad R(z) := (|z_1|, \dots, |z_n|).$$

It can be viewed as the map of the space $\mathbb{C}^n$ to its quotient $\mathbb{R}^n_{\geq 0}$ by the torus action.

Proposition 2.7 *Consider a given series $\sum_k c_k z^k$ in variable $z = (z_1, \dots, z_n)$. Let Ω denote its convergence domain. For every $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{\geq 0}$ set*

$$\phi(r) := \overline{\lim}_{k\to\infty} (|c_k| r^k)^{\frac{1}{|k|}}, \quad r^k = r_1^{k_1} \dots r_n^{k_n}. \tag{2.1}$$

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be a point with $z_j \neq 0$ for all j.
1) One has $z \in \Omega$, if and only if $\phi(R(z)) < 1$.
2) One has $z \in \partial\Omega$, if and only if $\phi(R(z)) = 1$.

In the proof of the proposition we use following homogeneity and continuity properties of the upper limit function $\phi(r)$.

Proposition 2.8 *1) One has*

$$\phi(\lambda r) = \lambda \phi(r) \text{ for every } \lambda > 0. \tag{2.2}$$

2) The function $\phi(r)$ is well-defined and continuous on the set $\mathbb{R}^n_+$ of vectors with positive components, provided it is well-defined for at least one $r \in \mathbb{R}^n_+$.

Proof Formula (2.2) follows from definition. Let us prove Statement 2). Let the upper limit $\phi(r)$ exist for at least one $r \in \mathbb{R}^n_+$. Then it exists for every other $r' \in \mathbb{R}^n_+$ with $r' \leq r$, by definition: $(r')^k \leq r^k$ for every $k \in \mathbb{Z}^n_{\geq 0}$. Then $\phi(r)$ extends from the set of the latter r' to all of $\mathbb{R}^n_+$ by homogeneity formula (2.2). Let us prove that $\phi(r)$ is continuous on $\mathbb{R}^n_+$.

Claim 1. *Let $r = (r_1, \dots, r_n)$ with $r_j > 0$ for all j, and let $\lambda > 1$. For every $r' \in \mathbb{R}^n_+$ with $\lambda^{-1} r < r' < \lambda r$ one has*

$$\lambda^{-1}\phi(r) < \phi(r') < \lambda\phi(r). \tag{2.3}$$

Proof One has $\lambda^{-1}(r^k)^{\frac{1}{|k|}} < (r')^k < \lambda(r^k)^{\frac{1}{|k|}}$, by definition. This together with the definition of the value $\phi(r)$ implies (2.3). □

Claim 1 immediately implies continuity of the function ϕ at every $r \in \mathbb{R}^n_+$ (taking λ arbitrarily close to 1) and finishes the proof of Proposition 2.8. □

Proof of Proposition 2.7. Let $z \in \mathbb{C}^n$ be a point with $z_j \neq 0$ for all j. Let us prove the first statement of Proposition 2.7: $z \in \Omega$ if and only if $\phi(R(z)) < 1$. Indeed, set $r = R(z)$. Let $z \in \Omega$. Then $\lambda z \in \Omega$ for some $\lambda > 1$ (openness; let us fix this λ). Hence, for every $\lambda' \in (1, \lambda)$ the series $\sum_k c_k(\lambda' z)^k$ converges, and thus, its terms are uniformly bounded in k. This implies that the upper limit of the $|k|$-th roots of its terms is no greater than 1. Thus, $\phi(r) \leq (\lambda')^{-1} < 1$. Conversely, let $\phi(r) < 1$. Fix a $\lambda > 1$ such that $\phi(r) < \lambda^{-1}$. Then $\phi(\lambda r) = \lambda\phi(r) < 1$. In other words, $\overline{\lim}_{k\to\infty}(|c_k|(\lambda r)^k)^{\frac{1}{|k|}} < 1$. This implies that the expression under limit is less than one, thus $|c_k|(\lambda r)^k < 1$, whenever $|k|$ is large enough. Therefore, $\Delta_{\lambda r} \subset \Omega$, by Abel's Lemma, and thus, $z \in \Delta_{\lambda r} \subset \Omega$. This proves Statement 1) of Proposition 2.7.

Let us prove Statement 2). Let $r = R(z)$ and let $\phi(r) = 1$. Then for every $\lambda \in (0, 1)$ one has $\phi(\lambda r) < 1$, and hence, $\Delta_{\lambda r} \subset \Omega$ (Statement 1)). This implies that $z \in \overline{\Omega}$. But $z \notin \Omega$, since $\phi(R(z)) = 1$ and by Statement 1). Hence, $z \in \partial\Omega$. Conversely, let $z \in \partial\Omega$ and $R(z) \in \mathbb{R}^n_+$. Then z is the limit of a convergent sequence $w_m \to z$, $w_m \in \Omega$, $R(w_m) \in \mathbb{R}^n_+$, and $\phi(R(w_m)) < 1$, by Statement 1). Therefore, $\phi(R(z)) \leq 1$, by continuity. We know that $\phi(R(z))$ cannot be less than 1, since $z \notin \Omega$ and by Statement 1). Therefore, $\phi(R(z)) = 1$. Proposition 2.7 is proved. □

Now let us prove that every holomorphic function is C^∞-smooth, using the fact that it is locally the sum of a converging power series. We show that the latter is its Taylor series.

The higher derivatives $\frac{\partial^l f}{\partial z^l}$, $\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l}$ of function of one variable and the higher derivatives

$$\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l} = \frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}}, \ k, l \in \mathbb{Z}_{\geq 0}$$

of a function of n complex variables are defined by subsequent differentiations. They are independent on the choice of order of differentiations (if the order of smoothness of the function is no less than the number of differentiations). This follows from the general fact that every two differential operators with constant coefficients commute.

Example 2.9 Let $f(z) = z_1^{s_1} \dots z_n^{s_n}$. Then

$$\frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}} = 0 \text{ whenever } l \neq 0;$$

$$\frac{\partial^k f}{\partial z^k} = 0 \text{ whenever } k_j > s_j \text{ for a certain } j;$$

$$\frac{\partial^k f}{\partial z^k} = \prod_{j=1}^{n} \frac{s_j!}{(s_j - k_j)!} z^{s-k}, \text{ whenever } k_j \leq s_j \text{ for all } j.$$

Remark 2.10 All the above statements on power series remain valid for power series $\sum_k c_k (z-p)^k$ with arbitrary $p \in \mathbb{C}^n$: the convergence domain is a union of polydisks centered at p, etc.

Proposition 2.11 *Let a power series $f(z) = \sum_k c_k z^k$ has a non-empty convergence domain. Then its sum $f(z)$ is holomorphic and C^∞-smooth there and*

$$c_0 = f(0), \ c_k = \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|} f}{\partial z^k}(p), \tag{2.4}$$

Proof Without loss of generality we consider that $p = 0$. The convergence domain is a union of convergence polydisks. Fix a convergence polydisk Δ_r and let us prove the above regularity statements in Δ_r. We claim that each derivative (of any order) of the series $\sum_k c_k z^k$ converges uniformly on compact subsets in Δ_r. Let $\phi(r)$, $\phi_1(r)$ denote respectively the upper limits (2.1) corresponding to the initial series and its derivative

$$\frac{\partial}{\partial z_1} (\sum_k c_k z^k) = \sum_k k_1 z_1^{-1} c_k z^k.$$

One has

$$\phi_1(r) = \overline{\lim}_{k\to\infty}((|k_1 r_1^{-1} c_k|r^k)^{\frac{1}{|k|-1}} \leq \overline{\lim}_{k\to\infty}(|c_k|r^k)^{\frac{1}{|k|-1}} = \phi(r) \leq 1.$$

Thus, the above derivative series converge uniformly on compact subsets in Δ_r, by Proposition 2.7. For higher derivatives the proof is analogous: the l-th derivation yields a new multiplier polynomial in k of fixed degree $|l|$, and its contribution to the above upper limit cancels out after taking a root of order $|k|$, as in the above inequality. This implies infinite differentiability of the function f, and each its partial derivative is equal to the sum of the corresponding derivative series. In particular, $\frac{\partial f}{\partial \bar{z}_j} = 0$, since this holds for each term of the power series. Hence, f is holomorphic. The value $\frac{\partial^{|k|} f}{\partial z^k}(0)$ is equal to the free term of the corresponding derivative series, i.e., $k_1! \dots k_n! c_k$. This proves (2.4) and the proposition. □

Corollary 2.12 *A function f on a domain $V \subset \mathbb{C}^n$ is holomorphic, if and only if each point $p \in V$ has a neighborhood where f is a sum of a converging power series $\sum_k c_k (z-p)^k$. The coefficients c_k are given by formula (2.4).*

The corollary follows from the above proposition and Theorem 1.11.

3 Analytic extension. Erasing singularities. Hartogs Theorem

Theorem 3.1 *(Uniqueness of analytic extension). Every two holomorphic functions on a connected domain $\Omega \subset \mathbb{C}^n$ that are equal on an open subset coincide on all of Ω.*

Proof It is sufficient to show that if a holomorphic function f on a connected domain Ω vanishes on some open subset $V \subset \Omega$, then $f \equiv 0$ on all of Ω. To do this, let us consider the subset

$$K = \cap_{k \in (\mathbb{Z}_{\geq 0})^n} \{ \frac{\partial^{|k|} f}{\partial z^k} = 0 \} \subset \Omega : \ K \supset V.$$

One has $f|_K \equiv 0$, since the latter intersection includes $k = 0$. The subset $K \subset \Omega$ is closed, being an infinite intersection of closed subsets, since $f \in C^\infty(\Omega)$ (Corollary 2.12). The set K is open. Indeed, at each point $p \in K$ the function f has vanishing Taylor series coefficients, by definition and formula (2.4). Hence, $f \equiv 0$ on a neighborhood of the point p, and thus, the latter neighborhood is contained in K. Therefore, K is a nonempty closed and open subset of a connected domain Ω, hence $K = \Omega$ and $f \equiv 0$ on Ω. □

Proposition 3.2 (Openness Principle.) *Each non-constant holomorphic function on a connected domain is an open map: the image of each open subset is open.*

Proof Let f be a non-constant holomorphic function on a connected domain Ω. It suffices to show that for every point $z \in \Omega$ the image of arbitrary ball centered at z contains a neighborhood of the image $f(z)$. Fix a $z \in \Omega$ and a complex line L through z where $f|_L \not\equiv const$ in a neighborhood of z. The line L exists since f is locally non-constant (uniqueness of analytic extension). The restriction of the function f to a disk in $L \cap \Omega$ centered at z is an open map, being a non-constant holomorphic function of one complex variable. This implies that the image of every disk as above contains a neighborhood of the point $f(z)$, and hence, so does the image of arbitrary ball in Ω centered at z. The proposition is proved. □

Corollary 3.3 (Maximum Principle.) *The module of a non-constant holomorphic function on a connected domain Ω cannot achieve its maximum in Ω.*

Proof If a module of a holomorphic function $f \not\equiv const$ achieves its maximum at a point $z \in \Omega$, then the image $f(\Omega)$ contains the point $f(z)$ but avoids the exterior of the circle through $f(z)$ centered at 0. Hence, it contains no its neighborhood, – a contradiction to Openness Principle. The corollary is proved. □

Theorem 3.4 (Liouville). *Every bounded holomorphic function on all of $\mathbb{C}^n$ is constant.*

Proof The restriction of a bounded holomorphic function f to each complex line through the origin is constant, being a bounded holomorphic function on $\mathbb{C}$ (Liouville Theorem in one variable). Therefore, $f \equiv f(0)$ on $\mathbb{C}^n$. □

It is known that for every domain $V \subset \mathbb{C}$ there exists a holomorphic function on V that extends analytically to no point of its boundary. This statement is false in higher dimensions. A basic counterexample, the *Hartogs Figure* is provided by the next theorem.

Theorem 3.5 *(Hartogs) Let* $R = (R_1, \ldots, R_n)$, $R_j > 0$, $1 \leq k < n$, $r = (r_1, \ldots, r_k)$, $r_s < R_s$. *Set* $R^k = (R_1, \ldots, R_k)$, $R^{n-k} = (R_{k+1}, \ldots, R_n)$. *Let* $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ *be an open subset. Let* $z = (z_1, \ldots, z_n)$ *be coordinates on* $\mathbb{C}^n$. *Set* $t = (z_1, \ldots, z_k)$, $w = (z_{k+1}, \ldots, z_n)$,

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}}, \ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n, \ \Omega = A \cup B.$$

(In the case, when $n = 2$, $k = 1$, $V = D_{r_2}$, $r_2 < R_2$, *the domain* Ω *is the so-called* **Hartogs Figure,** *see Fig.1.) Then every function holomorphic on* Ω *extends holomorphically to the whole polydisk* $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.

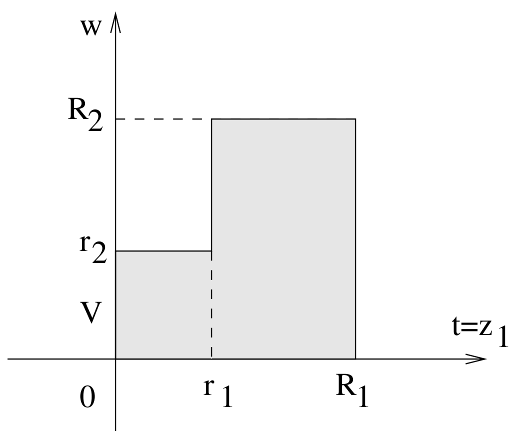

Figure 1: The Hartogs Figure for $n = 2$: picture in the positive quadrant

Proof For simplicity, let us prove the theorem in the case, when $n = 2$, $k = 1$: thus $R^k = R_1$, $R^{n-k} = R_2$, $z = (z_1, z_2)$, $t = z_1$, $w = z_2$. The proof in the general case is analogous. Let f be a function holomorphic on Ω. Fix an arbitrary $\delta \in (r_1, R_1)$. For every $w \in V$ the function $f(z_1, w)$ is holomorphic in $z_1 \in D_{R_1} \subset \mathbb{C}$, since $D_{R_1} \times \{w\} \subset B \subset \Omega$. Therefore, for every $z_1 \in D_\delta$ it is expressed as Cauchy integral

$$f(z_1, w) = \frac{1}{2\pi i} \oint_{|z_1|=\delta} \frac{f(\zeta, w)}{\zeta - z_1} d\zeta. \tag{3.1}$$

For every fixed $w \in D_{R_2}$ the subintegral function is holomorphic in $z_1 \in D_\delta$. Hence, the integral is also holomorphic in $z_1 \in D_\delta$, as in the proof of Osgood's Lemma. For every fixed $\zeta \in D_{R_1} \setminus D_\delta \supset \partial D_\delta$ the function $f(\zeta, w)$ is holomorphic in $w \in D_{R_2}$, since $\{\zeta\} \times D_{R_2} \subset A \subset \Omega$. Finally, the subintegral function is holomorphic in $(z_1, w) \in D_\delta \times D_{R_2}$, and hence, so is the integral. Thus, formula (3.1) extends the function $f(z_1, w)$ holomorphically to $D_\delta \times D_{R_2}$. This holomorphic extension is unique, by the Uniqueness Theorem for holomorphic extension. Thus, f is holomorphic there and hence, on all of $\Delta_R = D_{R_1} \times D_{R_2}$, since δ is an arbitrary number between r_1 and R_1. This proves the theorem for $n = 2$ and $k = 1$. Theorem 3.5 is proved. □

Exercise 3.6 (Seminar.) Prove Theorem 3.5 in the general case using multidimensional Cauchy integral.

Theorem 3.7 *(Erasing compact singularities). Let $G \subset \mathbb{C}^n$ be an open subset, $K \Subset G$ be a compact subset. Let both G and the complement $G \setminus K$ be connected. Then every function holomorphic on $G \setminus K$ extends holomorphically to all of G.*

We prove this theorem only in the case, when the ambient domain is a polydisk. Its proof in general case is more complicated and can be done by using, e.g., Bochner–Martinelli integral formula.

Proof of Theorem 3.7 in the case, when G is a polydisk. Let us prove the theorem in the case when $n = 2$: in higher dimensions the proof is literally analogous. Let $G = \Delta_R$, $R = (R_1, R_2)$. Let K_1, K_2 denote respectively the images of the compact set K under the projections to the z_1- and z_2-axes: $K_1 \Subset D_{R_1}$, $K_2 \Subset D_{R_2}$. Fix an open subset $V \subset D_{R_2} \setminus K_2$ and a $0 < r_1 < R_1$ such that $K_1 \Subset D_{r_1}$. Let Ω be the Hartogs figure from Theorem 3.5 constructed by the chosen r_1, V and R. One has $\Omega \subset \Delta_R \setminus K$. Therefore, every function holomorphic on $\Delta_R \setminus K$ is holomorphic on Ω, and hence, extends to a function holomorphic on all of Δ_R, by Theorem 3.5. □

Exercise 3.8 (Seminar). Prove that every function holomorphic on the complement of a polydisk centered at the origin to a coordinate subspace of codimension at least two extends holomorphically to the whole polydisk.

Hint. Construct an appropriate Hartogs figure in the complement to the coordinate subspace in question.

4 Implicit Function and Constant Rank Theorems. Complex manifolds. Extension theorems for functions on manifolds

4.1 Implicit Function and Constant Rank Theorems

Theorem 4.1 (Holomorphic Implicit Function Theorem) *Let $U \subset \mathbb{C}^n \times \mathbb{C}^\ell$, $(0,0) \in U$. Let $F : U \to \mathbb{C}^\ell$, $(X,Y) \mapsto F(X,Y)$ be a holomorphic map with $F(0,0) = 0$. Let the partial differential $\frac{\partial F}{\partial Y}(0) : T_0\mathbb{C}^\ell \to T_0\mathbb{C}^\ell$ be a non-degenerate linear operator. Then there exists a neighborhood $\Delta = V \times W$ of the origin in $\mathbb{C}^n \times \mathbb{C}^\ell$ such that the intersection $\Delta \cap \{F = 0\}$ is the graph $\{Y = Y(X)\}$ of a holomorphic mapping $Y : V \to W$. Its differential $dY(X_0)$ at each point X_0, set $Y_0 = Y(X_0)$, is equal to $-(\frac{\partial F}{\partial Y})^{-1}(X_0,Y_0)\frac{\partial F}{\partial X}(X_0,Y_0)dX$. That is, the latter matrix product is equal to the Jacobian matrix of the mapping $Y(X)$ at X_0.*

Proof The mapping F being considered as a real mapping of the domain $U \subset \mathbb{C}^n \times \mathbb{C}^\ell = \mathbb{R}^{2n} \times \mathbb{R}^{2\ell}$ to $\mathbb{C}^\ell = \mathbb{R}^{2\ell}$ satisfies the statement of the real Implicit Function Theorem from analysis. The above function $Y(X)$ is well-defined and C^1-smooth on some domain $V \subset \mathbb{C}^n$ containing the origin, and $Y(0) = 0$. The above formula for its derivative holds in terms of real linear operators. The derivatives of the map F in X and in Y are both $\mathbb{C}$-linear at each point $(X_0, Y_0) \in U$, by holomorphicity. Therefore, the differential $dY(X_0)$ is also $\mathbb{C}$-linear at each point $X_0 \in V$. But each C^1-smooth (vector) function on V with $\mathbb{C}$-linear differential at each point is holomorphic. Hence, $Y(X)$ is holomorphic on V. This proves the Holomorphic Implicit Function Theorem. □

Recall the following definition.

Definition 4.2 A mapping $F : U \to V$ of complex domains (manifolds) is *biholomorphic*, if it is holomorphic and has a holomorphic inverse.

Theorem 4.3 *(Holomorphic Inverse Map Theorem) Let $U \subset \mathbb{C}^n$ be a neighborhood of the origin. A holomorphic map $G : U \to \mathbb{C}^n$ with non-degenerate differential $dG(0)$ is always a biholomorphic map of some neighborhood of the origin onto an open subset in $\mathbb{C}^n$.*

Proof It suffices to apply the Implicit Function Theorem to the function $F(X,Y) = G(Y) - X$. □

Remark 4.4 Each biholomorphic mapping is always a C^∞ diffeomorphism. There exist no biholomorphic mappings of domains of different dimensions, since this is true for diffeomorphisms.

Theorem 4.5 (Constant Rank Theorem). *Let $U \subset \mathbb{C}^n$ be a neighborhood of the origin. Let $F : U \to \mathbb{C}^m$ be a holomorphic map, $F(0) = 0$. Let its differential have constant rank $\ell \leq m$ on U. Then there exist neighborhoods $V \subset U$, $W \subset \mathbb{C}^m$ of the origin and biholomorphisms (coordinate changes) $g : V \to V_1 \times V_2 \subset \mathbb{C}^{n-\ell}_z \times \mathbb{C}^\ell_w$, $h : W \to W_1 \times W_2 \subset \mathbb{C}^\ell_x \times \mathbb{C}^{m-\ell}_y$ such that $F(V) \subset W$ and $h \circ F \circ g(z, w) = (w, 0)$.*

Proof The proof of this theorem repeats the classical proof of the similar theorem from calculus. It is done in two steps.

Step 1: case, when $\ell = m$. Let us split coordinates in $\mathbb{C}^n$ into two groups (z, w), $z = (z_1, \dots, z_{n-\ell})$, $w = (w_1, \dots, w_\ell)$ so that the partial differential $\frac{\partial F(0,0)}{\partial w}$ is epimorphic, i.e., invertible. Consider the auxiliary mapping $H : (z, w) \mapsto (z, F(z, w))$. It is well-defined and holomorphic on a neighborhood $V_1 \times V_2 \subset \mathbb{C}^{n-\ell} \times \mathbb{C}^\ell$ of the origin. Its differential at the origin is non-degenerate, by construction. Therefore, shrinking the above domains V_1, V_2, we get that it is a biholomorphism of the product $V_1 \times V_2$ onto its image: a neighborhood V of the origin in $\mathbb{C}^\ell$. Hence, the mapping H has a holomorphic inverse of the form $g : (z, y) \mapsto (z, G(z, y))$. By construction, $F \circ g(z, y) = y$. The theorem is proved with $h = Id$.

Step 2: case, when $\ell < m$. Let (z, w) be the above splitting of the coordinates on $\mathbb{C}^n$. Let us split the coordinates in the image space $\mathbb{C}^m$ in two groups (x, y), $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_{m-\ell})$, so that the map $\widetilde{F} := x \circ F$ has rank ℓ at the origin (and hence, on some its neighborhood). Then applying Step 1 to the map $\widetilde{F}$ we get that there exists a map $g : V_1 \times V_2 \to \mathbb{C}^n$, $g(0) = 0$, such that $\widetilde{F} \circ g(z, w) = w$. Hence, $F \circ g(z, w) = (w, \psi(z, w))$, ψ is holomorphic on a neighborhood of the origin. Shrinking V_1 and V_2, we consider that the latter neighborhood coincides with $V_1 \times V_2$. . The rank of the latter map $F \circ g$ should coincide with the rank of the map F, that is, with the dimension ℓ of the w-variable. This implies that the function $\psi(z, w)$ has zero derivative in z and hence, depends only on w. Post-composing the map $F \circ g(z, w) = (w, \psi(w))$ with the map $h : (x, y) \mapsto (x, y - \psi(x))$ yields the map $(z, w) \mapsto (w, 0)$. The Constant Rank Theorem is proved. □

4.2 Comlex manifolds and extension of functions

Definition 4.6 A *complex manifold* of complex dimension d is a real $2d$-dimensional manifold M admitting an atlas where all the transition functions are biholomorphic. In more detail, it is a topological space M that admits a covering by open sets U_j such that there exist homeomorphisms $H_j : U_j \to V_j \subset \mathbb{C}^d$ with the following property:

- for every two intersected open subsets U_i and U_j the transition maps $H_j \circ H_i^{-1} : H_i(U_i \cap U_j) \to H_j(U_i \cap U_j) \subset V_j$ are holomorphic (they are biholomorphic, since their inverses $H_i \circ H_j^{-1}$ are also holomorphic by definition).

Here we suppose that M has a countable basis of neighborhoods.

Definition 4.7 A function $f : M \to \mathbb{C}$ on a complex manifold M is *holomorphic* if for every j the function $f \circ H_j^{-1} : V_j \to \mathbb{C}$ is holomorphic. A holomorphic map $M \to \mathbb{C}^n$ and a holomorphic map between complex manifolds are defined analogously.

Definition 4.8 Let M be a n-dimensional complex manifold, and let $k \in \mathbb{N}$, $k \leq n$. A subset $A \subset M$ is a k-dimensional *complex (holomorphic) submanifold*, if it is closed and each point $x \in A$ has a neighborhood $U = U(x) \subset M$ that admits a biholomorphism h on a neigborhood of the origin in $\mathbb{C}^n_{(z_1,\dots,z_n)}$, $h(x) = 0$, such that h sends the intersection $A \cap U$ onto the intersection of the image $h(U)$ with the coordinate k-plane $\{z_{k+1} = \dots = z_n = 0\}$. The tangent space of a submanifold at its point is defined in the same way, as the tangent space of a real submanifold; in the holomorphic case under consideration the tangent space has a natural structure of a complex vector space.

Example 4.9 Let $f : M \to \mathbb{C}^{n-k}$ be a holomorphic vector function, and let $A = \{f = 0\}$. Let 0 be not its critical value: the differential $df(x)$ at each point $x \in A$ is non-degenerate, that is, has rank k. Then A is a submanifold, which follows from the Implicit Function Theorem.

Theorem 4.10 *(Erasing* codim ≥ 2 *singularities). Let M be a complex manifold, and let $A \subset M$ be a complex submanifold of codimension at least two. Then every function holomorphic on $M \setminus A$ extends holomorphically to all of M.*

Proof It suffices to show that each point $x \in A$ has a neighborhood $U = U(x) \subset M$ such that each holomorphic function $f : U \setminus A \to \mathbb{C}$ extends holomorphically to all of U. This holds for a neighborhood U that admits a biholomorphism onto a polydisk so that $A \cap U$ is sent to a coordinate subspace of codimension at least two: see Exercise 3.8. □

5 Analytic sets

First in Subsection 5.1 we introduce notion of analytic subsets and present their basic properties. Then in Subsection 5.2, 5.3 we study the case of germs of local hypersurfaces, zero loci of germs of holomorphic functions, where we prove Weierstrass Preparation Theorem and factoriality of the local ring of holomorphic functions. Afterwards we will pass to the general theory of analytic subsets.

5.1 Introduction and main properties

Definition 5.1 An *analytic subset* in a complex manifold M is a subset $A \subset M$ such that each point $p \in A$ has a neighborhood $U = U(p) \subset M$ where there exists a finite collection of holomorphic functions $f_j : U \to \mathbb{C}$, $j \in J$, such that

$$A \cap U = \{f_j = 0 \mid j \in J\}.$$

Remark 5.2 Each analytic subset is closed. Any holomorphic submanifold is an analytic subset, but the converse is not true. For example, the coordinate cross $A = \{xy = 0\} \subset \mathbb{C}^2$ and the cusp curve $B = \{y^2 = x^3\} \subset \mathbb{C}^2$ are analytic subsets. But they are not submanifolds. See a brief explanation (with an exercise) below.

Definition 5.3 The *regular part* of an analytic subset $A \subset M$ is the subset A_{reg} consisting of those points $x \in A$ such that there exists a neighborhood $U = U(x) \subset M$ for which the intersection $U \cap A$ is a submanifold in U. This is an open subset in A. The complement $A_{sing} := A \setminus A_{reg}$ is a closed subset in M called the *singular part* of the set A.

Exercise 5.4 (Seminar). Let $U \subset \mathbb{C}^n$ be a domain. Consider a holomorphic function $f : U \to \mathbb{C}$. Set

$$Z_f := \{f = 0\} \subset U, \quad Z_f^o := \{x \in Z_f \mid df(x) \neq 0\}.$$

Let Z_f^o be dense in Z_f, and the complement $Z_f^s := Z_f \setminus Z_f^o$ be non-empty. Show that $Z_{f,sing} = Z_f^s$. Deduce the statements of the above remark.

Hint. Suppose to the contrary that Z_f is a local submanifold at a point $x \in Z_f^s$. Then there exists a neighborhood $W = W(x)$ and a biholomorphism

H that sends W to a domain $V \subset \mathbb{C}^n$ and sends $Z_f \cap W$ to a coordinate subspace of codimension 1, say, $z_n = 0$. Then the line L through $H(x)$ parallel to the z_n-axis intersects $H(W)$ once, and so does any close line. But one can find a parallel line L' arbitrary close to L that intersects $H(W)$ at least twice: the restriction to L of the function $f \circ H^{-1}$ has zero $H(x)$ of total multiplicity bigger than one, since its differential at $H(x)$ is zero.

Proposition 5.5 *Any analytic subset A in a connected manifold M either coincides with all of M, or is nowhere dense. In the latter case its complement is dense.*

Proof The interior $Int(A)$ is obviously open. It suffices to prove that it is closed: this will imply that it is either emply, or all of M, by connectivity. Let $p \in M$ be an accumulation point of $Int(A)$. Then $p \in A$. Hence, there exists a neighborhood $U = U(p) \subset M$ such that the functions f_j defining the set A in U are holomorphic in U and vanish on a non-empty open subset $Int(A) \cap U$. Hence, they vanish identically, by uniqueness of analytic extension. This implies that $A \cap U = U$ and hence, $p \in Int(A)$ and $Int(A)$ is closed. The proposition is proved. □

Exercise 5.6 (Seminar). Show that in the above second case the complement $M \setminus A$ is connected.

Exercise 5.7 (Seminar). A finite union of analytic subsets $A_1 \cup \dots A_k$ is analytic. A finite intersection of analytic subsets $A_1 \cap \dots \cap A_k$ is analytic.

Proof It suffices to prove these statements for union (intersection) of two analytic subsets A_1 and A_2 of a complex manifold M.

Let us show that $A_1 \cap A_2$ is analytic. Let $x \in A_1 \cap A_2$. Let U be its neighborhood in M where each $A_j \cap U$ is defined as zero locus of a finite collection F_j of holomorphic functions. Then $A_1 \cap A_2 \cap U$ is the zero locus of the functions from the finite collection $F_1 \cup F_2$. Therefore, $A_1 \cap A_2$ is analytic.

Let us now show that $A_1 \cup A_2$ is analytic. Recall that the sets A_1 and A_2 are closed, being analytic. In the case, when they are disjoint, there is nothing to prove: each point $x \in A_1 \cup A_2$ lies only in one subset A_j, its neighborhood $U \subset M$ small enough intersects A_j only, and thus, $U \cap (A_1 \cup A_2) = U \cap A_j$ is defined by the same collection of holomorphic functions, as A_j; hence it is analytic. Let now $x \in A_1 \cap A_2$. Let $U \subset M$ be its neighborhood where each $A_j \cap U$ is zero locus of a finite collections F_j of holomorphic functions on U. Then the zero locus of the products fg,

$f \in F_1$, $g \in F_2$, coincides with $U \cap (A_1 \cup A_2)$. Hence, $A_1 \cup A_2$ is analytic. The statements of the exercise are proved. □

Theorem 5.8 *The regular part of an analytic subset $A \subset M$ is dense in A.*

Proof We prove Theorem 5.8 by induction in the dimension n of the ambient manifold M.

Induction base: for $n = 1$ the statement of the theorem is obvious.

Induction step. Let the statement of the theorem be proved for $n \leq k$. Let us prove it for $n = k+1$. Let $x \in A$. Let us show that A_{reg} accumulates to x. Fix some small neighborhood $V = V(x) \subset M$ and consider that x is the origin in a holomorphic chart containing V. We show that $A_{reg} \cap V \neq \emptyset$ and then we apply this statement to arbitrarily small V. Thus, we deal with A as an analytic subset in $V \subset \mathbb{C}^n$, $0 \in A$, where A is defined as the zero locus of a finite collection of functions holomorphic on V. Fix a holomorphic function $f \not\equiv 0$ on V, $f|_A \equiv 0$. There exists a (higher) partial derivative g of the function f (which may be f itself) that vanishes identically on A and such that some of its partial derivatives $\frac{\partial g}{\partial z_j}$ does not vanish identically on A. Indeed in the opposite case all the partial derivatives of the function f would vanish at 0, and hence, $f \equiv 0$, - a contradiction. Thus, $g|_A \equiv 0$, and thus, the analytic subset $\Gamma = \{g = 0\} \subset V$ contains A. The regular part of the set Γ contains the open subset $\Gamma^0 := \Gamma \cap \{\frac{\partial g}{\partial z_j} \neq 0\} \subset \Gamma$, since $\frac{\partial g}{\partial z_j}|_A \not\equiv 0$ and by the Implicit Function Theorem. The intersection $A^0 := \Gamma^0 \cap A$ is non-empty, by assumption. On the other hand, it is an analytic subset in the complex manifold Γ^0 of dimension $n-1$. Hence, its regular part A^0_{reg} is dense in $\Gamma^0 \cap A$ (and thus, non-empty), by the induction hypothesis. But A^0_{reg} is contained in A_{reg}. Indeed, for every $y \in A^0_{reg}$ and every neighborhood $W = W(y) \subset V$ such that $W \cap \Gamma \subset \Gamma^0$ the intersection $W \cap \Gamma^0$ is obviously a submanifold in W, and $A \cap W = A^0_{reg} \cap W$ is a submanifold in the latter submanifold. Hence, $A \cap W$ is a submanifold in W: a submanifold of a submanifold in W is obviously a submanifold in W. Finally, A_{reg} contains a non-empty subset $A^0_{reg} \subset V$. Applying the above arguments to arbitrarily small neighhborhood V we get that A_{reg} accumulates to x. Hence, A_{reg} is dense. Theorem 5.8 is proved. □

Definition 5.9 The *dimension* $\dim_x A$ of an analytic set A at its regular point $x \in A_{reg}$ is the dimension at x of the submanifold $A \cap U$ in a small neighborhood $U = U(x)$. Its dimension at a point $x \in A_{sing}$ is

$$\dim_x A := \overline{\lim}_{y \in A_{reg},\ y \to x} \dim_y A.$$

The *dimension of analytic set* A is

$$\dim A := \sup_{x \in A} \dim_x A = \max_{x \in A_{reg}} \dim_x A.$$

Theorem 5.10 *(given without proof). For every analytic subset $A \subset M$ its singular part A_{sing} is an analytic subset in M of dimension strictly less than* $\dim A$.

Proposition 5.11 *The preimage of an analytic subset in a manifold N under a holomorphic mapping $M \to N$ is an analytic subset in M.*

The proposition obviously follows from definition.

Definition 5.12 A map $F : M \to N$ between topological spaces is *proper,* if the preimage of every compact subset in N is a compact subset in M.

A fundamental result of the theory is the following theorem

Theorem 5.13 (Remmert Proper Mapping Theorem). *Let M, N be complex manifolds, and let $A \subset M$ be an analytic subset. Let $F : M \to N$ be a holomorphic map whose restriction to A is proper. Then the image $F(A) \subset N$ is an analytic subset.*

Corollary 5.14 *Let M, N be complex manifolds, and let N be compact. Let $A \subset M \times N$ be an analytic subset. Then the projection of the set A to M is an analytic subset in M.*

Theorem 5.15 *(was it at the seminar?) Each holomorphic bounded function on a complement of a complex manifold to an analytic subset extends holomorphically to the whole ambient manifold.*

Proof It suffices to prove the local version of the theorem, for a bounded holomorphic function f on the complement of a domain $U \subset \mathbb{C}^n_{z_1,\dots,z_n}$ to the zero locus $Z_g = \{g = 0\}$ of a holomorphic function $g : U \to \mathbb{C}$. Fix an $x \in Z_g$. It suffices to show that f extends holomorphically to a neighborhood of the point x in Z_g. Passing to appropriate local chart we can and will consider that $x = 0$, set $z = (z_1, \dots, z_{n-1})$, $w = z_n$, and that g is holomorphic on the polydisk $\Delta = \Delta_r \times D_\delta \subset \mathbb{C}^{n-1}_z \times C_w$, and that for every fixed $z \in \Delta_r$ one has $g(z, w) \not\equiv 0$ in $w \in D_\delta$. Then one can find a circle $S^1 = \{|w| = s\} \subset D_\delta$ and $\sigma = (\sigma_1, \dots, \sigma_{n-1})$, $\sigma_j < r_j$, such that $\Delta_\sigma \times S^1$ is disjoint from the zero locus Z_g. The intersection of each disk $\{z\} \times D_\delta$ with Z_g is a discrete set of points. The function $f(z, w)$ with fixed z extends there holomorphically, being bounded (Erasing Singularity Theorem for holomorphic functions in one variable). Thus, for every $z \in \Delta_\sigma$ and every w with $|w| < s$ one has

$$f(z,w) = \frac{1}{2\pi i} \oint_{S^1} \frac{f(z,\eta)}{\eta - w} d\eta,$$

by Cauchy Formula. The subintegral expression is holomorphic in $(z,w) \in \Delta_\sigma \times D_s$. Therefore, the above Cauchy integral extends $f(z,w)$ holomorphically to the latter product, and hence, to the neighborhood $(\Delta_\sigma \times D_s) \cap Z_g$ of the point $x = 0$ in Z_g. The theorem is proved. □

5.2 Weierstrass polynomials. Weierstrass Preparatory Theorem

Definition 5.16 A polynomial $P_z(w) = w^d + a_1(z)w^{n-1} + \cdots + a_0(z)$ with variable coefficients depending holomorphically on $z = (z_1, \ldots, z_n)$ from a neighborhood of the origin in $\mathbb{C}^n$ with $a_j(0) = 0$ is a holomorphic function in $n+1$ variables (z,w) called a *Weierstrass polynomial* in w.

Remark 5.17 For every fixed z a Weierstrass polynomial does not vanish identically in w and has the same number d of roots with multiplicity.

Definition 5.18 Let $f(z,w)$ be a germ of holomorphic function at $(0,0)$ in $\mathbb{C}^n_z \times \mathbb{C}_w$, $f(0,0) = 0$, that does not vanish identically on the w-axis. Let $\delta > 0$, $r = (r_1, \ldots, r_n)$, $r_j > 0$ be such that the function f is holomorphic on $\Delta_r \times \overline{D}_\delta$, $f(0,w) \neq 0$ for $w \in \overline{D}_\delta \setminus \{0\}$ and $f|_{\Delta_r \times \partial D_\delta} \neq 0$. Then $\Delta := \Delta_r \times D_\delta$ is called a *Weierstrass polydisc* for the function f.

Remark 5.19 If $f(0,w) \not\equiv 0$ on a neighborhood of zero in the w-axis, then a Weierstrass polydisk always exists. In general, if f is a holomorphic function on a neighborhood of the origin in $\mathbb{C}^{n+1}$, then one can choose coordinates $(z_1, \ldots, z_n, w)$ in such a way that $f(0,w) \not\equiv 0$, and hence, in the latter coordinates a Weierstrass polydisk exists.

Theorem 5.20 (Weierstrass preparatory theorem). *Let $f(z,w)$ be a holomorphic function on a neighborhood of the origin in $\mathbb{C}^{n+1} = \mathbb{C}^n_z \times \mathbb{C}_w$, $z = (z_1, \ldots, z_n)$, with $f(0,0) = 0$ and $f(0,w) \not\equiv 0$. Let $\Delta = \Delta_r \times D_\delta$ be a Weierstrass polydisk. Then there exists a unique Weierstrass polynomial $P_z(w)$ such that on some neighborhood U of the origin one has $f(z,w) = h(z,w)P_z(w)$, $h(z,w)$ is a holomorphic function on the latter neighborhood U, $h(0,0) \neq 0$. Moreover, $P_z(w)$ is holomorphic on $\Delta_r \times \mathbb{C}_w$ and $h(z,w)$ is holomorphic and nonvanishing on $\Delta_r \times \overline{D}_\delta$.*

Proof Fix a Weierstrass polydisc $\Delta = \Delta_r \times D_\delta$. Set $g_z(w) = f(z,w)$. The function g_0 has geometrically unique zero in $\overline{D}_\delta$: the origin. Let d denote its multiplicity. Then for every $z \in \Delta_r$ the function g_z has d roots with multiplicities in D_δ and does not vanish on its boundary. Let $b_1(z), \dots, b_d(z)$ denote its roots. The coefficients of the Weierstrass polynomial we are looking for are uniquely determined as the basic symmetric polynomials $\sigma_s = \sigma_s(z)$ in $b_j(z)$ up to sign. (This already proves the uniqueness.) They vanish at $z = 0$ by assumption. Let us show that they are holomorphic functions in z. Indeed, they are expressed as polynomials in the power sums $\hat{\sigma}_s(z) = \sum_j b_j^s(z)$, $s \in \mathbb{N}$. One has

$$\hat{\sigma}_s(z) = \frac{1}{2\pi i} \oint_{\partial D_\delta} \frac{\zeta^s \frac{\partial f}{\partial w}(z,\zeta)}{f(z,\zeta)} d\zeta. \tag{5.1}$$

Indeed, the latter integral is equal to the sum of residues of the subintegral expression. The nonzero residues may exist only at those ζ, where $g_z(\zeta) = f(z,\zeta) = 0$. The residue value corresponding to a root ζ of the function $g_z(w)$ of multiplicity ν is equal to $\nu\zeta^s$. Indeed, one has

$$g_z(u) = f(z,u) = c(u-\zeta)^\nu(1 + O(u-\zeta)), \text{ as } u \to \zeta; \ c \neq 0,$$

$$\frac{\partial f}{\partial w}(z,u) = c\nu(u-\zeta)^{\nu-1}(1+o(1)) + O((u-\zeta)^\nu) = \frac{\nu}{u-\zeta} f(z,u)(1+o(1)).$$

This implies that the residue at ζ is equal to $\nu\zeta^s$. This proves (5.1). The right-hand side in (5.1) is holomorphic in $z \in \Delta_r$, since the subintegral expression is holomorphic and its restriction to the integration circle is a uniformly bounded function whenever z run over arbitrary compact subset in Δ_r. Therefore, the integral and hence, the power sums $\hat{\sigma}_s(z)$ are holomorphic on Δ_r. Hence, the elementary symmetric polynomials σ_s are also holomorphic. Therefore, the function

$$P_z(w) = \prod_{j=1}^{d}(w - b_j(z)) = w^d + \sum_{s=1}^{d}(-1)^s \sigma_s(z) w^{d-s}$$

is a Weierstrass polynomial vanishing exactly on the zero set $\Gamma = \{f = 0\}$ of the function f. The ratio $h = \frac{f}{P}$ and its inverse h^{-1} are holomorphic functions on the complement $(\Delta_r \times \overline{D}_\delta) \setminus \Gamma$. Let us show that each of them extends holomorphically to Γ (say h; for h^{-1} the proof is the same): then the theorem follows immediately. For every fixed z the function $h(z,w)$ has a nonzero limit, as w tends to a root of the polynomial $P_z(w)$, since the latter root has the same multiplicity for both functions $P_z(w)$ and $g_z(w)$. Therefore, the function $h(z,w)$ is holomorphic in $w \in \overline{D}_\delta$ for every fixed $z \in \Delta_r$. Hence, it can be written as Cauchy integral

$$h(z,w) = \frac{1}{2\pi i} \oint_{|\zeta|=\delta} \frac{h(z,\zeta)}{\zeta - w} d\zeta, \ w \in D_\delta.$$

The subintegral expression is holomorphic in $(z,w) \in \Delta$ and uniformly bounded with derivatives and continuous on compact subsets in Δ. Therefore, the latter integral, and hence h are holomorphic there. Similarly, h^{-1} is holomorphic. Hence, h is a unity. □

5.3 Local rings. Factorization of holomorphic functions as products of irreducible ones

Definition 5.21 Let X be a topological space, $x \in X$. Two functions f and g defined on neighborhoods U_f and U_g of the point x are called *x-equivalent,* if there exists a neighborhood $W = W(x)$ where $f \equiv g$. The *germ* of a function at a point x is its x-equivalence class.

Remark 5.22 In general, two functions (e.g., smooth functions on a manifold) defining the same germ at x can be distinct. But *if two holomorphic functions on a connected manifold M have the same germ at some point, then they are identically equal on M,* by uniqueness of analytic extension. For every point $x \in M$ there is a 1-to-1 correspondence between germs at x of functions f holomorphic on some its neighborhoods U_f depending on f (i.e., functions holomorphic just at x) and germs of holomorphic functions at $0 \in \mathbb{C}^n$, $n = \dim M$, or equivalently, converging power series.

Definition 5.23 The ring of germs of holomorphic functions f at $0 \in \mathbb{C}^n$ will be called the *local ring* and denoted $\mathcal{O}_n$. Recall that a *unity* of a ring is an invertible element, i.e., an element u for which there exists an inverse u^{-1}, $uu^{-1} = 1$. Thus, a unity in $\mathcal{O}_n$ is a germ of holomorphic function that does not vanish at 0.

Remark 5.24 The Weierstrass Preparation Theorem implies that *each germ of holomorphic function $f(z,w)$ at $(0,0) \in \mathbb{C}^n_z \times \mathbb{C}_w$ with $f(0,0) = 0$ and $f(0,w) \not\equiv 0$ is the product of a Weierstrass polynomial and a unity in $\mathcal{O}_{n+1}$.*

Definition 5.25 An element of a ring is *irreducible,* if it is not a unity and cannot be presented as a product ab, where a and b are not unities. A ring is *factorial,* if each its non-zero element that is not a unity can be represented in a unique way (up to permutation and multiplication by unities) as a product of powers of irreducible elements times a unity.

Here we prove the following theorem.

Theorem 5.26 *The local ring $\mathcal{O}_n$ is factorial.*

In the proof of Theorem 5.26 we use the Weierstrass Preparatory Theorem and the following well-known Gauss Lemma and property of Weierstrass polynomials.

Lemma 5.27 *(Gauss). Let R be a factorial ring. Then the polynomial ring $R[w]$ is also factorial.*