CRITICAL POINTS, 22.02

V.A. VASSILIEV

1. Self-intersection index of the vanishing cycle of a Morse singularity

We know that for a Morse singularity in n variables the Milnor fiber V_ζ (ζ non-critical) is diffeomorphic to the space of the tangent bundle of S^{n-1}. It is an oriented $(2n-2)$-dimensional manifold, therefore the self-intersection index of the basic element Δ of $\tilde{H}_{n-1}(V_\zeta)$ (called its *vanishing cycle*) is well-defined. For instance, if $n = 2$ then V_ζ is diffeomorphic to a cylinder, and hence the self-intersection index is equal to zero. Moreover, we know that any even-dimensional hedgehog can be shaved, therefore these self-intersection indices are equal to 0 for all even n. (This fact has also a more general proof: the self-intersection index of any odd-dimensional cycle is equal to zero by the skew commutativity of intersection indices.) We will check in classes that for odd n this index (with respect to the complex orientation) is equal to $2(-1)^{n(n-1)/2}$.

2. Monodromy operator of an isolated singularity

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity; $B_\varepsilon \subset \mathbb{C}^n$ and $D_\delta \subset \mathbb{C}$ the same objects as previously. For any $\zeta \in D_\delta$, the corresponding Milnor fiber V_ζ is defined as $f^{-1}(\zeta) \cap B_\varepsilon$. The sets V_ζ form a locally trivial fiber bundle over the punctured disc $D^*_\delta \equiv D_\delta \setminus \{0\}$. Any element

α of the group $\pi_1(D^*_\delta, \zeta_0) \simeq \mathbb{Z}$ defines the corresponding *monodromy operator* $M_\alpha \in \mathrm{Aut}(\tilde{H}_{n-1}(V_{\zeta_0}))$. (Covering homotopy...)

On significance in PDE's and integral geometry...

Example 1. $n = 1$, $f = z^2$, $\tilde{H}_0(V_\zeta) \simeq \mathbb{Z}$. For $\zeta = \delta$ this group is generated by the cycle $\{\sqrt{\delta}\} - \{-\sqrt{\delta}\}$. Monodromy: ζ runs the circumference $\{\delta e^{it}\}$, $t \in [0, 2\pi]$. Two points $z(t) = \pm\sqrt{\delta} \cdot e^{it/2}$ run a half-circumference each, and hence permute when t runs the entire segment $[0, 2\pi]$. So, the corresponding monodromy operator is the multiplication by -1.

Example 2. $n = 1$, $f = z^k$, $\tilde{H}_0(V_\zeta) \simeq \mathbb{Z}^{k-1}$. Monodromy operator is... to be calculated in classes. Clearly, it is an unipotent operator of order k.

Example 3. n arbitrary, f Morse: $f = z_1^2 + \cdots + z_n^2$. $\tilde{H}_{n-1}(V_\zeta) = \mathbb{Z}$. Basic monodromy operator is equal to $(-1)^n$.

Indeed, consider real coordinates x_j, y_j so that $z_j = x_j + iy_j$. The group $\tilde{H}_{n-1}(V_\delta)$ is generated by the class of (somehow oriented) sphere of radius $\sqrt{\delta}$ in the plane $\{y_1 = \cdots = y_n = 0\}$ with coordinates $x_1, \ldots, x_n$. The covering homotopy can be realized by operators $T_t : (z_1, \ldots, z_n) \mapsto e^{it/2}(z_1, \ldots, z_n)$, $V_\delta \mapsto V_{e^{it}\delta}$. For $t = 2\pi$ this map sends any point of the sphere $\{x_1^2 + \cdots + x_n^2 = 1\}$ to the opposite point in $\mathbb{R}^n$. The central symmetry in $\mathbb{R}^n$ changes the orientation of the sphere S^{n-1} if and only if n is odd.

Proposition. If f is homogeneous of degree d then monodromy operator is unipotent of degree d.

Proof. Take an arbitrary cycle in V_ζ and act on it by the family of rotations T_t : $(z_1, \dots, z_n) \mapsto e^{it/d}(z_1, \dots, z_n)$.

Exercise. If f is quasihomogeneous of degree d with weights $a_1, \dots, a_n$, then basic monodromy operator is unipotent of degree ? .

Exercise. If $F(z_1, \dots, z_n, w_1, \dots, w_m) \equiv f(z_1, \dots, z_n) + g(w_1, \dots, w_m)$, then

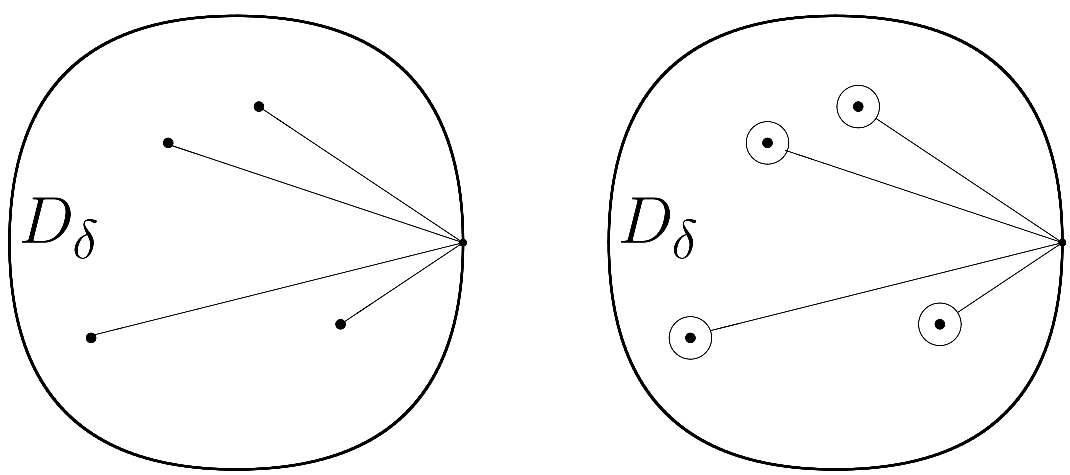
$$\tilde{H}_{n+m-1}(V_\delta(F)) \simeq \tilde{H}_{n-1}(V_\delta(f)) \otimes \tilde{H}_{m-1}(V_\delta(g)),$$

and the monodromy operator for F is the tensor product of monodromy operators for f and g.

3. Monodromy group

Let $\tilde{f}$ be a very small strict Morsification of f, so that all $\mu(f)$ critical values of $\tilde{f}$ in points of B_ε are close to the center of D_δ. Define $\tilde V_\zeta \equiv \tilde f^{-1} \cap B_\varepsilon$.

The group $\tilde H_{n-1}(\tilde V_\zeta)$ for a non-critical value $\zeta \in D_\delta$ is again isomorphic to $\mathbb{Z}^{\mu(f)}$ for $\zeta \in D_\delta$ a non-critical value of $\tilde f$; it is generated by $\mu(f)$ vanishing cycles defined by a system of non-intersecting paths in D_δ connecting ζ with all critical values. Also, there is a natural identification between groups $\tilde H_{n-1}(\tilde V_\zeta)$ and $\tilde H_{n-1}(V_\zeta)$ if ζ is far away from all critical values, say $\zeta = \delta$.



The group $\pi_1(D_\delta \setminus \{\mu \text{ critical values of } \tilde f\}, \delta)$ is free with $\mu(f)$ generators corresponding to critical values (and specified by distinguished paths): *simple loops* or *pinches*.

It acts on the group $\tilde{H}_{n-1}(\tilde{V}_\delta)$; this action (or its image) is called the *monodromy group* of morsification $\tilde{f}$.

The monodromy operator of f commutes with the action of the element of the group $\pi_1(D_\delta \setminus \{\mu \text{ critical values }\}, \delta)$ defined by the δ-circle. So to calculate it we can study all actions of particular pinches and then to take the composition of these actions corresponding to μ pinches, the product of which is equivalent to the circle.

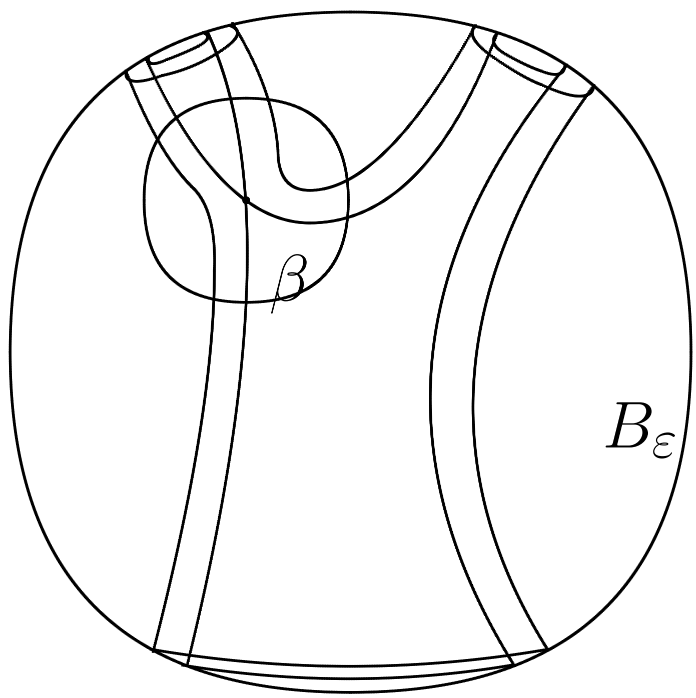
4. Picard–Lefschetz formula

The study of these μ particular operators is based on a local consideration at corresponding Morse critical points. Namely, consider the operator M_j acting on $\tilde{H}_{n-1}(V_\delta)$ and defined by the pinch of the j-th critical value and corresponding path. Let $\Delta_j \in \tilde{H}_{n-1}(V_\delta)$ be the basic cycle vanishing along this path.

Theorem. Operator M_j sends an arbitrary element $\Delta \in \tilde{H}_{n-1}(\tilde{V}_\delta)$ to

$$\Delta + (-1)^{n(n+1)/2}\langle \Delta, \Delta_j \rangle \Delta_j.$$

.



Take again the small ball β around the corresponding critical point a_j; let ζ_j be the end of segments in the j-th pinch (which is very close to the corresponding critical value ξ_j)

How does the small circle around ξ_j act on $\tilde{H}_{n-1}(V_{\zeta_j})$? I.e., given a cycle $\bar{\Delta} \in \tilde{H}_{n-1}(V_{\zeta_j})$, how does this operator change $\bar{\Delta}$?

Localization principle. This action depends only on the local behavior of $\bar{\Delta}$ inside β, namely on the class of $\bar{\Delta}$ in the relative homology group $H_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta)$. Also, it adds to Δ a cycle contained inside β, i.e. represented by an element of $\tilde{H}_{n-1}(V_{\zeta_j} \cap \beta)$.

Remark. We know that $\tilde{H}_{n-1}(V_{\zeta_j} \cap \beta)$ is equal to $\mathbb{Z}$ as $V_{\zeta_j} \cap \beta$ is the Milnor fiber of a Morse singularity. Therefore the Poincare dual relative group $H_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta)$ also is isomorphic to $\mathbb{Z}$.

Proposition. There is an operator (called local variation operator)

$$Var_j : H_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta) \to \tilde{H}_{n-1}(V_{\zeta_j} \cap \beta)$$

such that the action of the j-th circle on $\tilde{H}_{n-1}(V_{\zeta_j})$ is equal to the sum of identity operator and the composition of three operators:

1) the reduction modulo the complement of β,

2) variation operator Var_j, and

3) the obvious operator $\tilde{H}_{n-1}(V_{\zeta_j} \cap \beta) \to \tilde{H}_{n-1}(V_{\zeta_j})$ induced by the inclusion of these varieties.

Picard–Lefschetz formula is an explicit expression of this operator.

We will describe this operator in the next lecture (in the case of arbitrary singularities, and not only Morse ones).

Example. Suppose that the intersection of $\bar{\Delta}$ with the ball β is empty, i.e. $\bar{\Delta} \subset V_{\zeta_j} \setminus \beta$. Then the action of our j-th circle (going around ξ_j) maps $\bar{\Delta}$ to itself. Indeed, the sets $V_\zeta \setminus \beta$ form a locally trivial (and hence trivializable) fiber bundle over entire neighborhood of the critical point ξ_j.

An arbitrary cycle $\bar{\Delta} \subset V_{\zeta_j}$ is homologous to a cycle equal to the sum of a chain in $V_{\zeta_j} \setminus \beta$ and several times the standard generator of the group $\tilde{H}_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta) \sim \mathbb{Z}$. Then we can apply the covering homotopy to this chain in such a way that its part lying outside β will come exactly to itself (which implies in particular that the boundary of the "interior" part $\bar{\Delta} \cap \beta$ also will come to itself) and the difference between the result and the initial part inside β will be an absolute cycle in $V_{\zeta_j} \cap \beta$: the result of our variation operator.

This operation

1) can be described in local terms, i.e. depends only on the class of $\bar{\Delta}$ in $\tilde{H}_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta)$ and commutes with equivalence of singularities, and

2) is linear.

By (complex) Morse lemma it should be expressed by a single number depending only on n: the standard generator ∇_j of the group $\tilde{H}_{n-1}(V_{\zeta_j}, V_{\zeta_j} \setminus \beta)$ goes to $C(n)$ times the standard generator Δ_j of $\tilde{H}_{n-1}(V_{\zeta_j} \cap \beta)$.

Remark. In fact, generators of these groups are defined up to the choice of orientations. However, we can agree to choose these orientations in such a way that the

(Poincare) intersection index $\langle \nabla_j, \Delta_j \rangle$ is equal to 1. Then, changing the orientation of one of these cycles we change also the other one, and hence our formula remains stable.

Theorem. (Picard–Lefschetz formula). $C(n) = (-1)^{n(n+1)/2}$, i.e. the local variation operator of a Morse singularity maps a generator of the group $\tilde{H}_{n-1}(V_\zeta, \partial V_\zeta)$ to $(-1)^{n(n+1)/2}$ times the corresponding generator of $\tilde{H}_{n-1}(V_\lambda \cap \beta)$.

Example. Let n be odd. By §1 the reduction operator sends Δ to $2(-1)^{(n-1)/2}\nabla$, hence $M(\Delta) = \Delta + 2(-1)^{(n-1)/2}C(n)\Delta$. But by Example 3 we have $M(\Delta) = -\Delta$. Therefore $C(n) = (-1)^{(n+1)/2}$ for odd n.