CRITICAL POINTS, 22.02

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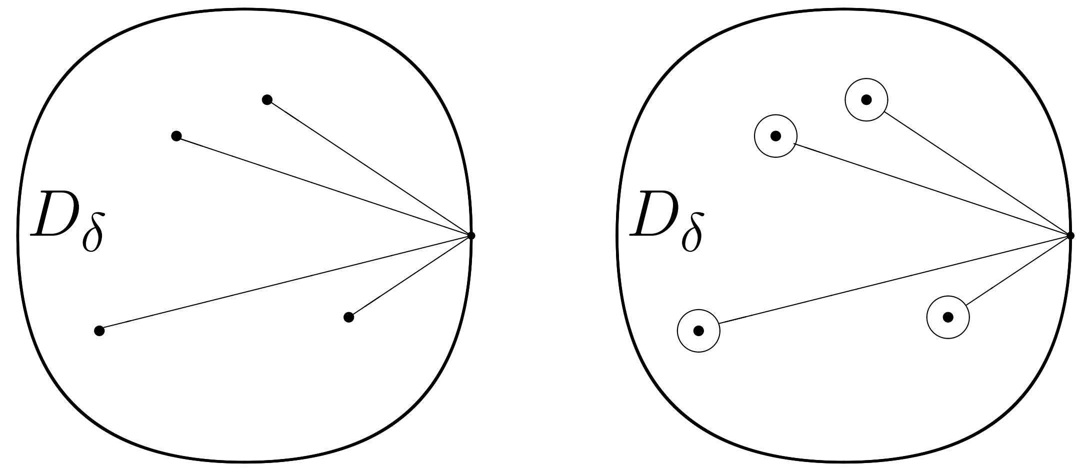
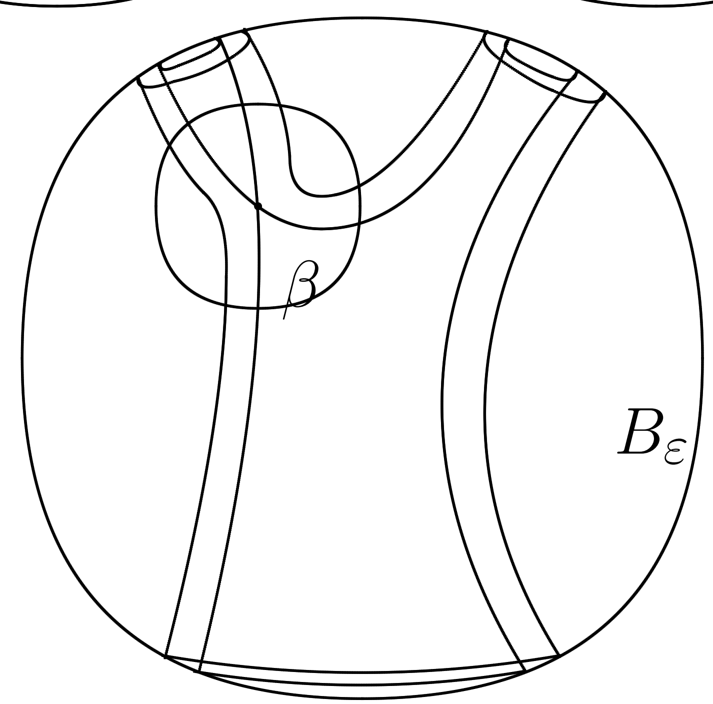
1. Variation operators





1.1. **Reminders.** $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^1, 0)$, $B_\varepsilon \subset \mathbb{C}^n$, $D_\delta \subset \mathbb{C}^1$.

Milnor fiber $V_\zeta = f^{-1}(\zeta) \cap B_\varepsilon$ for $\zeta \in D_\delta$.

$\tilde{H}_{n-1}(V_\zeta) \sim \mathbb{Z}^{\mu(f)}$ (Milnor number).

$\tilde{f}$ a perturbation of f, $\tilde{V}_\zeta = \tilde{f}^{-1}(\zeta) \cap B_\varepsilon$.

$c(\tilde{f}) = \{$critical values of $\tilde{f}\}$. Sets $\tilde{V}_\zeta$ form a locally trivial bundle over $D_\delta \setminus c(\tilde{f})$, and sets $\partial \tilde{V}_\zeta$ over entire D_δ (so the latter bundle can be trivialized). $\pi_1(D_\delta \setminus c(\tilde{f}), \delta)$ acts on $H_*(\tilde{V}_\delta)$: monodromy group.

1.2. **Operators Var$_\gamma$.** γ is a loop in $D_\delta \setminus c(\tilde{f})$ with basepoint δ (or its class in π_1). Var_γ is an operator

$$\tilde{H}_{n-1}(\tilde{V}_\delta, \partial \tilde{V}_\delta) \to \tilde{H}_{n-1}(\tilde{V}_\delta).$$

Take a relative cycle ∇ in $(\tilde{V}_\delta, \partial \tilde{V}_\delta)$. Move it over the points of loop γ in such a way that $\partial\nabla$ moves in correspondence with the fixed trivialization of the fiber bundle over D_δ with fibers $\partial\tilde{V}_\zeta$. Then the result of this movement, $\tilde{\nabla}$, is a relative cycle with the same boundary, and $\tilde{\nabla} - \nabla$ is an absolute cycle in $\tilde{V}_\delta$. By definition, its homology class in $\tilde{H}_{n-1}(\tilde{V}_\delta)$ is $\mathrm{Var}_\gamma(\{\nabla\})$. It depends only on the homology class of ∇ and the class of the loop γ in $\pi_1(D_\delta \setminus c(\tilde{f}))$.

Example. Suppose that class ∇ can be represented by an absolute cycle, i.e. is the image of a class $\Delta \in \tilde{H}_{n-1}(\tilde{V}_\delta)$ under the reduction homomorphism

$$J : \tilde{H}_{n-1}(\tilde{V}_\delta) \to \tilde{H}_{n-1}(\tilde{V}_\delta, \partial\tilde{V}_\delta).$$

Then

$$\mathrm{Var}_\gamma(J(\Delta)) = M_\gamma(\Delta) - \Delta$$

where M_γ is the monodromy operator.

1.3. **Picard-Lefschetz formula.** If $\tilde{f}$ is a morsification and γ is the simple loop corresponding to the j-th critical value, then

$$\mathrm{Var}_\gamma(\nabla) = (-1)^{n(n+1)/2}\langle \nabla, \Delta_j\rangle \Delta_j.$$

(Proof: consideration for Morse case.)

Corollary. For an absolute cycle $\Delta \in H_{n-1}(V_\delta)$,

$$M_{\gamma_j}(\Delta) = \Delta + (-1)^{n(n+1)/2}\langle \Delta, \Delta_j\rangle \Delta_j.$$

1.4. **Variation operator.** Variation operators can be defined for an arbitrary perturbation $\tilde{f}$ of f (not necessarily a Morsification), in particular for the function f itself. In this case $\pi_1(D_\delta \setminus c(f)) \sim \mathbb{Z}$, and we have a single operator.

Theorem. This operator $\mathrm{Var} : \mathbb{Z}^\mu \to \mathbb{Z}^\mu$ is an isomorphism.

Proof 1. Replace f by morsification $\tilde{f}$. Choose a system of non-intersecting paths from δ to critical values, and corresponding basis of vanishing cycles Δ_j in $\tilde{H}_{n-1}(\tilde{V}_\delta)$; let ∇_j be the Poincare dual basis in $\tilde{H}_{n-1}(\tilde{V}_\delta, \partial\tilde{V}_\delta)$, and γ_j corresponding simple loops. The basic circle is the composition

$$\gamma_\mu \circ \cdots \circ \gamma_2 \circ \gamma_1 \ .$$

Lemma. $\mathrm{Var}_{\gamma_2\circ\gamma_1}(\nabla) =$
$\mathrm{Var}_{\gamma_1}(\nabla) + \mathrm{Var}_{\gamma_2}(\nabla) + \mathrm{Var}_{\gamma_2}(J(\mathrm{Var}_{\gamma_1}(\nabla)))$.

Now it is easy to check that in these bases the variation operator will be triangular with numbers $(-1)^{n(n+1)/2}$ on the diagonal.

Proof 2 (**Milnor fibration on** S_ε^{2n-1}**).**
Definition: Knot $K(f)$ is the set $\partial B_\varepsilon \cap f^{-1}(0)$.

Consider the map

$$A : S_\varepsilon^{2n-1} \setminus K(f) \to S^1$$

defined by argument of f.

Lemma. This is a fiber bundle, with fibers homeomorphic to $V_\zeta \backslash \partial V_\zeta$, $\zeta \in D_\delta \backslash 0$. The closures of all these fibers (obtained by adding their common boundary $K(f)$) are homeomorphic to V_ζ.

Indeed, consider the vector field grad $|f|$ in B_ε, and act by it on V_ζ. Consider the map $V_\zeta \to S_\varepsilon^{2n-1}$ sending any point of V_ζ to the end of the trajectory of this vector field (at which it leaves the ball B_ε). This map defines a homeomorphism between V_ζ and the set of points in S_ε^{2n-1} at which f has the same argument as ζ, and $|f|$ is $\geq |\zeta|$.

Now, consider in S_ε^{2n-1} the closed subset

$$T = S_\varepsilon^{2n-1} \cap f^{-1}([0, +\infty)) \equiv A^{-1}(1) \cup K(f).$$

It is homeomorphic to V_δ. The exact sequence of the pair $(S_\varepsilon^{2n-1}, T)$:

$$\cdots \to \tilde{H}_{k+1}(S_\varepsilon^{2n-1}) \to \tilde{H}_{k+1}(S_\varepsilon^{2n-1}, T) \to \\ \to \tilde{H}_k(T) \to \tilde{H}_k(S_\varepsilon^{2n-1}) \to \cdots$$

$$\tilde{H}_{k+1}(S_\varepsilon^{2n-1}, T) = \bar{H}_{k+1}(S_\varepsilon^{2n-1} \setminus T).$$

$S_\varepsilon^{2n-1} \setminus T$ is fibered over $(0, 2\pi)$ with fiber $\sim V_\delta \setminus \partial V_\delta$.

By Künneth isomorphism,
$\tilde{H}_{k+1}(S_\varepsilon^{2n-1}, T) \equiv \bar{H}_{k+1}((V_\delta \setminus \partial V_\delta) \times (0, 2\pi)) \simeq \bar{H}_k(V_\delta \setminus \partial V_\delta) \equiv \tilde{H}_k(V_\delta, \partial V_\delta) \sim \mathbb{Z}^\mu$.

For $k = n-1$, the differential of our exact sequence is by construction equal to the variation operator. But it should be an isomorphism since $\tilde{H}_*(S_\varepsilon^{2n-1}) = 0$ in dimensions n and $n-1$.