CRITICAL POINTS, 22.03

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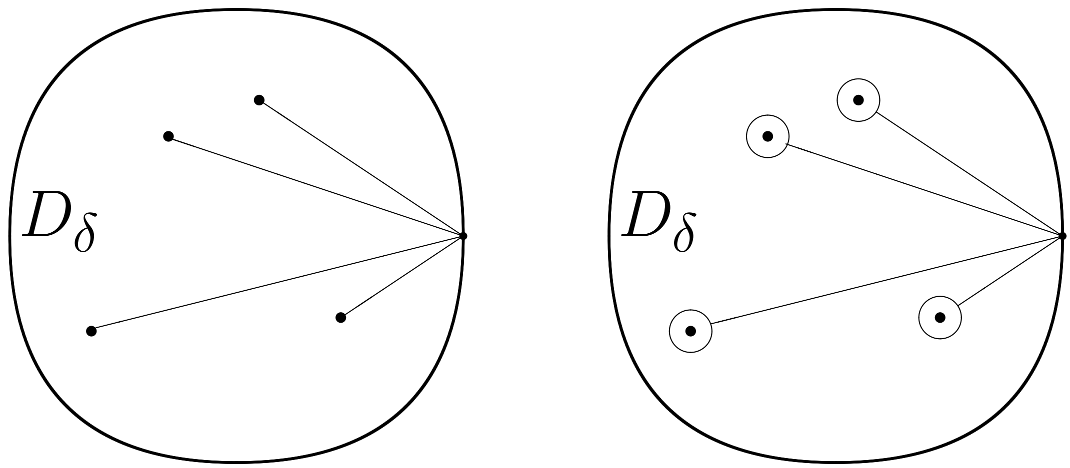
1. Intersection forms of singularities

So, monodromy group and variation operators are controlled by Picard–Lefschetz formulas

$$\mathrm{Var}_{\gamma_j}(\nabla) = (-1)^{n(n+1)/2}\langle \nabla, \Delta_j\rangle \Delta_j,$$

$$M_{\gamma_j}(\Delta) = \Delta + (-1)^{n(n+1)/2}\langle \Delta, \Delta_j\rangle \Delta_j,$$

where $\Delta \in \tilde{H}_{n-1}(V_\delta)$, $\nabla \in \tilde{H}_{n-1}(V_\delta, \partial V_\delta)$.



This makes the study of intersection forms in $\tilde{H}_{n-1}(V_\zeta)$ very important. Intersection matrix: $\mu(f) \times \mu(f)$ with entries $\langle \Delta_i, \Delta_j \rangle$ (for any basis of vanishing cycles).

This form is symmetric $\langle \Delta, \Delta' \rangle = \langle \Delta', \Delta \rangle$ if n is odd, and anti-symmetric $\langle \Delta, \Delta' \rangle = -\langle \Delta', \Delta \rangle$ if n is even. In particular, $\langle \Delta, \Delta \rangle = 0$ for any Δ if n is even. For odd n, the self-intersection index of any vanishing cycle is equal to 2 if $n \equiv 1 \mathrm{mod}\ 4$, and to -2 if $n \equiv 3 \mathrm{mod}\ 4$.

Theorem. Let n be odd. The quadratic form $\langle\Delta,\Delta\rangle$ in $\mathbb{Z}^{\mu(f)}$ is definite if and only if f is a simple singularity.

Indeed, for all fencing singularities it already is not definite. And if $f \prec \tilde{f}$ then the corresponding space $\mathbb{Z}^{\mu(\tilde{f})}$ is naturally included into $\mathbb{Z}^{\mu(f)}$.

Monodromy group as reflection group

If n is odd, then the operator M_{γ_j} works as reflection group in the group $\mathbb{Z}^{\mu(f)}$ supplied with the bilinear form $\langle\Delta,\Delta'\rangle$: this operator keeps fixed the elements "orthogonal" to Δ_j with respect to this form, and maps Δ_j to minus itself.

Reflection groups corresponding to simple singularities are the eponymous Weyl groups (extremely important in study of simple Lie algebras).

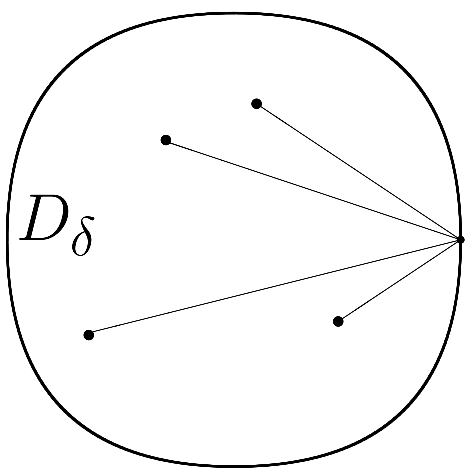
Exercise. Calculate the action of γ_j^{-1}.

Answer:

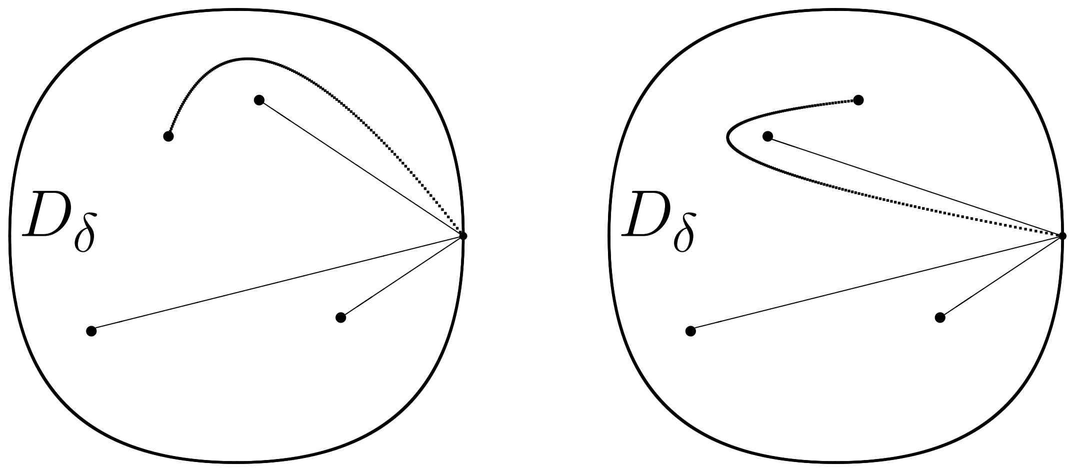
$$\Delta \mapsto \Delta + (-1)^{(n+1)(n+2)/2} \langle \Delta, \Delta_j \rangle \Delta_j.$$

Proof: in classes.

1.1. **Change of bases.**



Vanishing cycles $\Delta_j \in \tilde{H}_{n-1}(V_\delta)$ are numbered in the order of the arguments of tangents of paths leaving the non-critical value.



Then after the first (left hand) change the new basic cycles are as follows:

$$\check{\Delta}_2 = \Delta_1, \quad \check{\Delta}_1 = M_{\gamma_1^{-1}}(\Delta_2).$$

After the right-hand move:

$$\check{\Delta}_1 = \Delta_2, \quad \check{\Delta}_2 = M_{\gamma_1}(\Delta_2).$$

1.2. **Stabilization. Definition**. Two real function singularities, f and g are *stable equivalent* if they become equivalent after summation with non-degenerate quadratic forms in additional variables. For example, $f(x)+y^2$ and $f(x)$, $f(x)+y^2$ and $f(x)-y^2$, but $f(x)$, $x \in \mathbb{R}^n$, is not stable equivalent to $f(x)$ considered as a function in $\mathbb{R}^{n+1}$ not depending on the last coordinate.

For any singularity $f : \mathbb{R}^n \to \mathbb{R}$, the class of functions in n variables stable equivalent to f consists of rank $(f) + 1$ equivalence classes.

Definition for complex functions is analogous, but easier (no difference between $+$ and $-$).

Let $f(x) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a function with isolated singularity, and $f(x) + y^2$ be its stabilization. Let $\tilde{f}$ be a small Morsification of f, then $\tilde{f} + y^2$ is a small Morsification of $f + y^2$, with the same set of critical values. In particular, there is a 1–1 correspondence between the sets of vanishing cycles Δ_j of $\tilde{f}$ and $\check{\Delta}_j$ $\tilde{f} + y^2$ defined by one and the same system of paths (up to choice of orientations).

Theorem. For an appropriate choice of orientations of vanishing cycles, the intersection matrices of morsifications $\tilde{f}$ and $\tilde{f}+y^2$ are compatible:

$$\langle \check{\Delta}_i, \check{\Delta}_j \rangle = (-1)^n \langle \Delta_i, \Delta_j \rangle$$

if $i > j$, and

$$(-1)^{n+1} \langle \Delta_i, \Delta_j \rangle$$

if $i < j$.

In particular, double stabilization multiplies intersection matrix by -1.

1.3. **Dynkin diagram.** This is a convenient representation of intersection matrices (which are usually sparse). By stabilization formulas, we can assume that f depends on $n \equiv 3 (\mathrm{mod}\ 4)$ variables. Then Dynkin diagram of $\tilde{f}$ is a graph with $\mu(f)$ vertices, the vertices i and j are connected by $\langle \Delta_i, \Delta_j \rangle$

usual edges if this number is positive, and by $-\langle \Delta_i, \Delta_j \rangle$ dashed edges if it is negative.

Example. $n = 1$, $f \in A_k$, $f(x) = x^{k+1}$. Then in some basis of vanishing cycles $\Delta_1, \dots, \Delta_k$, the intersection indices are given by $\langle \Delta_i, \Delta_i \rangle = 2$, $\langle \Delta_i, \Delta_j \rangle = -1$ if $|i-j| = 1$, and $\langle \Delta_i, \Delta_j \rangle = 0$ if $|i-j| > 1$.

(In classes).

1.4. **Intersection matrices for direct sums of singularities.** Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be two isolated function singularities. Let $\tilde{f}$ and $\tilde{g}$ their small morsifications, and $\Delta_1, \dots, \Delta_{\mu(f)}$, $\check{\Delta}_1, \dots, \check{\Delta}_{\mu(g)}$ be corresponding systems of vanishing cycles defined by certain systems of paths. Then the group $\tilde{H}_{n+m-1}(V_\delta)$ (where $V(\delta)$ is the Milnor fiber of $f(x) + g(y)$) has a basis of vanishing cycles $\Delta_{(i,j)}$, $i \in \{1, \dots, \mu(f)\}$, $j \in \{1, \dots, \mu(g)\}$, and their intersection indices can be expressed through intersection indices of vanishing cycles of f and g. Namely,

Theorem (A.M.Gabrielov)

$$\langle \Delta_{(a,b)}, \Delta_{(a,d)} \rangle = \mathrm{sgn}(d-b)^n (-1)^{nm+\frac{n(n-1)}{2}} \langle \check{\Delta}_b, \check{\Delta}_d \rangle$$
if $b \neq d$;

$$\langle \Delta_{(a,b)}, \Delta_{(c,b)} \rangle = \mathrm{sgn}(c-a)^m (-1)^{mn+\frac{m(m-1)}{2}} \langle \Delta_a, \Delta_c \rangle$$
if $a \neq c$;

$$\langle \Delta_{(a,b)}, \Delta_{(c,d)} \rangle = 0$$
if $\mathrm{sign}(c-a)\mathrm{sgn}(d-b) = -1$;

$$\langle \Delta_{(a,b)}, \Delta_{(c,d)} \rangle = \mathrm{sgn}(c-a)(-1)^{mn} \langle \Delta_a, \Delta_c \rangle \langle \check{\Delta}_b, \check{\Delta}_d \rangle$$
if $\mathrm{sgn}(c-a)\mathrm{sign}(d-b) = 1$.

Exercise: to understand these formulas and formulate them in terms of Dynkin graphs.