CRITICAL POINTS, 29.03

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1. Gusein–Zade — A.'Campo method for calculating Dynkin diagrams of singularities of corank 2

Essential part of such a singularity is a function in 2 variables.

Let $f : (\mathbb{C}^2, \mathbb{R}^2, 0) \to (\mathbb{C}, \mathbb{R}, 0)$ be a *real* function singularity with finite Milnor number $\mu(f)$. A small perturbation $\tilde{f}$ of f is called a *sabirization* if it is also real (i.e. takes real values on $\mathbb{R}^2$), and has exactly $\mu(f)$ real critical points: all critical values at saddlepoints are equal to 0, at minima are negative, and at maxima are positive.

Let c_0, c_-, and c_+ be the numbers of critical points of these types.

The set $\tilde{f}^{-1}(0) \cap B_\varepsilon \cap \mathbb{R}^2$ is a curve with c_0 crossing points, and its complement in $B_\varepsilon \cap \mathbb{R}^2$ consists of $c_- + c_+$ bounded (i.e. separated from ∂B_ε) domains, each of which contains exactly one extremum point. Any two such domains separated by a piece of the curve $\tilde{f}^{-1}$ contain extrema of different types.

(Certainly, it is not a strict morsification if $c_0 > 1$).

Do all isolated real singularities in two variables have sabirizations? Probably yes, but this is not proved (TBMK). At least, any class of isolated singularities in two variables (not necessarily real) has a real form, which has a sabirization.

More precisely:

For any isolated singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ the set $f^{-1}(0)$ consists in a neighbourhood of $0 \in \mathbb{C}^2$ of several irreducible components.

Definition. For arbitrary isolated function singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, the $\mu =$ const stratum of f is the connected component of the set of singularities with the same Milnor number as f in the space of all function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with singularity at 0.

Proposition. All singularities from a $\mu =$ const stratum have equal Dynkin diagrams.

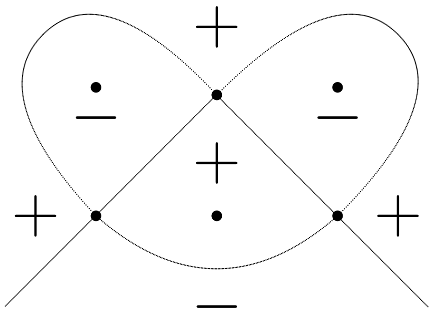
Definition. A real singularity $f : (\mathbb{C}^2, \mathbb{R}^2, 0) \to (\mathbb{C}, \mathbb{R}, 0)$ is completely real, if each local irreducible components of the complex curve

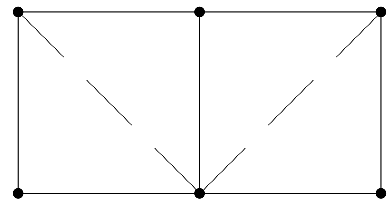
$f^{-1}(0)$ has 1-dimensional intersection with $\mathbb{R}^2$.

Proposition. Any $\mu =$ const stratum of isolated singularities in $\mathbb{C}^2$ contains a completely real representative.

Proposition (S.M.Gusein-Zade, N.A'Campo). Any completely real function singularity in 2 variables has a sabirization.

Theorem (S.M.Gusein-Zade, N.A'Campo). Let f be a real function singularity, and $\tilde{f}$ be some its sabirization. Then the singularity $f(x, y) + z^2$ has a Dynkin diagram (defined by a system of paths of a morsification arbitrarily close to perturbation $\tilde{f}$) the vertices of which are in one to one correspondence with the $c_- + c_+$ bounded domains and c_0 crossing points of $\tilde{f}$, and the edges are as follows:



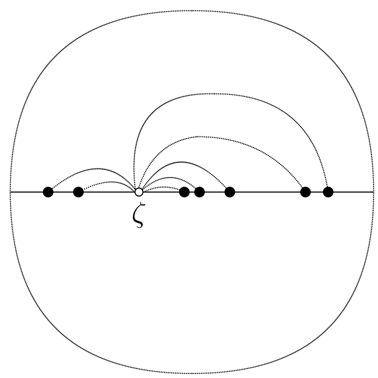


1) vertices corresponding to critical points of the same type are not connected by edges

2) two vertices, one of which corresponds to a saddlepoint, and the other one to an extremum, are connected by ordinary edges, the number of which (equal to 0, 1 or 2) is the number of separatrices of grad $\tilde{f}$ going from this saddlepoint to the domain containing this extremum;

3) two vertices, one of which corresponds to a minimum and the other one to a maximum are connected by dashed edges, the number of which is equal to the number of smooth segments of the curve $\tilde{f}^{-1}(0)$ separating them.

Proof. Let $\bar{f}$ be an arbitrary very small strict real morsification of $\tilde{f}$. Its critical points are then in a natural 1-1 correspondence with critical points of $\tilde{f}$ and have the same types.



For the non-critical value ζ take a negative real number slightly below all critical values at saddlepoints and above all values at minima. Then the vanishing cycles in $\tilde{V}_\zeta$ coming from minima are just the real contours going along the boundaries of corresponding bounded components of the set $\{\tilde{f} \leq 0\}$ and oriented as the boundaries of these domains (which have the fixed orientation of $\mathbb{R}^2$).

All of them are disjoint from one another, in particular their intersection indices are equal to 0.

The cycles vanishing in saddlepoints can be realized as follows. Let in local coordinates at such a point $\bar{f}$ be equal to $x^2 - y^2 + \alpha$, where $(\partial/\partial x, \partial/\partial y)$ is a positive frame in $\mathbb{R}^2$. Then for the vanishing cycle in $\tilde{V}_\zeta$ take the circle Δ given parametrically by

$$x(t) = \sqrt{\alpha - \zeta}\, i \sin t, y(t) = \sqrt{\alpha - \zeta}\, \cos t.$$

It intersects the real plane $\mathbb{R}^2$ at exactly two points corresponding to $t = 0$ and $t = \pi$. All these circles are located at corresponding saddlepoints, so they also are disjoint. Also, this proves (after a thorough accounting of orientations) the assertion on intersection indices of cycles vanishing at minima and saddlepoints.

To handle cycles vanishing at maxima, choose a value η slightly above all values at saddlepoints, and for any of them choose the vanishing cycle Δ' in $\tilde{V}_\eta$ by

$$x(t) = \sqrt{\eta - \alpha}\cos t, y(t) = \sqrt{\eta - \alpha}\, i \sin t.$$

The groups $H_1(\tilde{V}_\zeta)$ and $H_1(\tilde{V}_\eta)$ are identified by a small arc in positive plane connecting points ζ and η. This identification preserves intersection indices, and moves above described vanishing cycles Δ' of saddlepoints into corresponding cycles Δ. This proves (after accounting orientations and stabilization) the statement on intersection indices of cycles vanishing in saddlepoints and maxima.

It remains to prove statement 3.

Complex conjugation. Let $\tilde{f}$ be a real perturbation of f, and ζ a real non-critical value of $\tilde{f}$.

The complex conjugation acts in $\tilde{H}_{n-1}(\tilde{V}_\zeta)$. It is an involution, hence the space $\tilde{H}_{n-1}(\tilde{V}_\zeta, \mathbb{Q})$ splits into two subspaces H_+ and H_-, on which this involution acts respectively as multiplication by 1 and -1.

Proposition. If n is even, then subspaces H_+ and H_- are isotropic under the intersection index (i.e. this index of any two elements of one of these subspaces is equal to 0). If n is odd, then these two subspaces are orthogonal with respect to this intersection form.

Proof...

This gives another proof of the first statement of Theorem (that intersection indices of cycles of equal types are equal to 0).

Let again ζ be a real non-critical value below critical values at saddlepoints and above all minima of $\bar{f}$. Let Δ be a cycle in $H_1(\tilde{V}_\zeta)$ vanishing in a maximum along a path going in the upper half-plane. Generally it does not belong to H_+ or H_-. However $\Delta + \bar{\Delta}$ belongs to H_+, here $\bar{\Delta}$ vanishes over the complex conjugate path in the lower half-plane. In particular, for any cycle $\Delta_1 \in H_1(\tilde{V}_\zeta)$ vanishing in a minimum point we have $\langle \Delta_1, \Delta + \bar{\Delta} \rangle = 0$. But $\bar{\Delta}$ is obtained from Δ by Picard-Lefschetz monodromy around all critical values at saddlepoints. Therefore $\langle \Delta_1, \Delta \rangle$ is equal to half the number of saddlepoints of $\tilde{f}$ lying simultaneously on the boundary of domains corresponding to Δ_1 and Δ.