Exercises in Algebraic Geometry 16.09.2021

These exercises are due by September 23rd. This is a general rule: the due date is one week after the assignment. The final grade for the course is calculated as 0.1 of the percentage of completely solved exercises. You may submit e.g. the high quality scans of your handwritten solutions in the natural order. I will grade neither poor quality scans nor randomly ordered scans. You may also submit your handwritten solutions as a hardcopy or solutions typeset in TeX.

1. Prove that $\mathbb{A}^1$ is an affine algebraic variety.

2. We define *algebraic varieties* as topological spaces ringed with a sheaf of regular functions that can be covered by finitely many open *affine algebraic varieties*. Introduce a structure of algebraic variety on $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{\infty\}$ (with an open covering by two affine lines) (you have to define the topology and the sheaf of regular functions).

3. Consider a subvariety $Y \subset \mathbb{A}^2$ cut out by the equation $x^2 - y^3 = 0$. Equip it with the Zariski topology (closed subsets = finite subsets) and with a sheaf of regular functions $\mathcal{O}_Y(Y \setminus \{y_1, \ldots, y_n\})$ = rational functions on Y with possible poles in $\{y_1, \ldots, y_n\}$.
a) Construct a bijective morphism $\varphi\colon \mathbb{A}^1 \to Y$.
b) Prove that φ is a homeomorphism but not an isomorphism of algebraic varieties.

4. Let $\operatorname{char} \mathsf{k} = p$ (e.g. $\mathsf{k} = \overline{\mathbb{F}}_p$). We define the Frobenius morphism $\mathrm{Fr}\colon \mathbb{A}^1 \to \mathbb{A}^1$ by the formula $z \mapsto z^p$. Prove that Fr is a homeomorphism but not an isomorphism.

5. Let A be a commutative unital ring. Given $f \in A$ we define the *localization* $A_f := A[z]/(1 - fz)$.
a) Prove that A_f is the set of fractions $\frac{a}{f^n}$ modulo the following equivalence relation: $\frac{a}{f^n} = \frac{b}{f^m}$ iff for some N we have $f^{N+m}a = f^{N+n}b$.
b) Describe the kernel ideal of the natural morphism $\iota\colon A \to A_f$.
c) Describe the conditions on A, f when $A_f = 0$.
d) Prove that if A has no nilpotents (resp. no zero divisors), then A_f has no nilpotents (resp. no zero divisors) either.

Exercises in Algebraic Geometry 23.09.2021

1. Let A be a finitely generated commutative unital k-algebra without nilpotents, and $0 \neq f \in A$. Show that the localized algebra A_f is canonically isomorphic to the algebra of regular functions on the basic open subset $D_f \subset \operatorname{Spec}(A)$.

2. Let $X = \operatorname{Spec}(A)$ be an affine algebraic variety. Prove that
a) the basic open subsets $D_f \subset X,\ f \in A$, form a basis of the Zariski topology on X.
b) $D_f \cap D_g = D_{fg}$.

3. Prove that a) any algebraic variety is quasicompact (any open cover has a finite subcover).

b) A point of an algebraic variety is a closed subset.

4. Let $f(x,y)$ be an irreducible quadratic polynomial. Prove that the closed subvariety $\{f=0\}\subset\mathbb{A}^2$ is isomorphic either to $\mathbb{A}^1$ or to $\mathbb{A}^1\setminus\{0\}$.

5. Prove that the product of Zariski topologies on $\mathbb{A}^1$ is *not* equal to the Zariski topology on $\mathbb{A}^2$.

Exercises in Algebraic Geometry 30.09.2021

1. Let $\mathcal{A}$ be a finitely generated commutative unital k-algebra. Prove *E. Noether normalization lemma*: there is a subalgebra $\mathsf{k}[x_1,\dots,x_d]\subset\mathcal{A}$ such that $\mathcal{A}$ is a $\mathsf{k}[x_1,\dots,x_d]$-module of finite type (i.e. the $\mathsf{k}[x_1,\dots,x_d]$-algebra $\mathcal{A}$ is finite). In case $\mathcal{A}$ has no nilpotents, the corresponding morphism $\mathrm{Spec}(\mathcal{A})\to\mathbb{A}^d$ is called finite.

2. Let $J\subset A$ be an ideal in a commutative ring A. We set $S=1+J\subset A$; then $S\cdot S\subset S$, and $S+J\subset S$.

a) Prove *Nakayama lemma*: For a finite type A-module M such that $JM=M$, there is $s\in S$ such that $sM=0$.

b) Let A be a local ring (i.e. it has a unique maximal ideal $\mathfrak{m}\subset A$), and M a finite type A-module such that $\mathfrak{m}M=M$. Prove that $M=0$.

c) Let A be a local ring with the maximal ideal $\mathfrak{m}$. Let $\varphi\colon M\to N$ be a morphism of finite type A-modules such that $\varphi \pmod{\mathfrak{m}}\colon M/\mathfrak{m}M\to N/\mathfrak{m}N$ is surjective. Prove that φ is surjective.

3. Let T be an endomorphism of a finite type A-module M. Assume that T is epimorphic.

a) Prove that T is invertible.

b) Prove that T is invertible on any T-invariant A-submodule $N\subset M$.

4. Let T be an endomorphism of a finite type A-module M.

a) Deduce from the Cayley-Hamilton identity that there is a monic polynomial $P\in A[t]$ such that $P(T)=0$.

b) Prove that if the image $T(M)$ lies in $JM\subset M$ for an ideal $J\subset A$, then one can choose P to be of the form $P(t)=t^n+a_{n-1}t^{n-1}+\ldots+a_0$, where all the coefficients $a_i\in J$.

c) Prove Nakayama lemma (once again).

5. Let $\varphi\colon B\to A$ be a finite morphism of finitely generated commutative unital k-algebras without nilpotents (i.e. A is a finite B-algebra).

a) Prove that the induced morphism $\psi\colon\mathrm{Spec}(A)\to\mathrm{Spec}(B)$ is closed (i.e. the image of a closed subset is closed) and has finite fibers.

b) Prove that if φ is injective, then ψ is surjective.

Exercises in Algebraic Geometry 07.10.2021

1. Let $X = \bigcup_{i=1}^{N} U_i$ be an affine open cover of an algebraic variety X, such that all the intersections $U_{ij} = U_i \cap U_j$ are also affine: $U_i = \mathrm{Spec}(A_i)$, $U_{ij} = \mathrm{Spec}(A_{ij})$. Prove that $\mathsf{k}[X] = \mathrm{Ker}(\oplus_i A_i \to \oplus_{i<j} A_{ij})$ (collections of elements of A_i compatible under localizations $A_i \to A_{ij} \leftarrow A_j$).

2. a) Let $H_i \subset \mathbb{A}^n$ be a hyperplane $x_i = 0$. Find $\mathsf{k}[\mathbb{A}^n \setminus H_i]$.
b) Find $\mathsf{k}[\mathbb{A}^n \setminus \{0\}]$.

3. a) Let $\pi\colon X \to Y$ be a surjective map from an algebraic variety to a set Y. Then we can equip Y with the quotient topology and the following sheaf of regular functions: f on $U \subset Y$ is called regular if $\pi^* f$ is regular on $\pi^{-1}(U) \subset X$. In case $X = \mathbb{A}^n \setminus \{0\}$, and $Y = \mathbb{P}^{n-1} := X/\mathsf{k}^\times$ is the set of lines in $\mathbb{A}^n$ passing through the origin, prove that we obtain the structure of an algebraic variety on Y (i.e. produce an open cover by affine algebraic subvarieties of $\mathbb{P}^{n-1}$) — this is the projective space $\mathbb{P}^{n-1}$.
b) Find $\mathsf{k}[\mathbb{P}^{n-1}]$.

4. Given a homogeneous degree d polynomial $f \in \mathsf{k}[x_1, \ldots, x_n] = \mathsf{k}[\mathbb{A}^n]$ let $V_f \subset \mathbb{P}^{n-1}$ be the image of the basic open subspace $D_f \subset \mathbb{A}^n$ (with the origin removed). Prove that
a) The open sets $V_f \subset \mathbb{P}^{n-1}$ form a base of the Zariski topology on $\mathbb{P}^{n-1}$.
b) Each open set V_f is an affine algebraic variety; namely, $V_f = \mathrm{Spec}(\mathsf{k}[x_1, \ldots, x_n]^0_f)$, where $\mathsf{k}[x_1, \ldots, x_n]^0_f$ stands for the subalgebra of degree 0 elements in the localization $\mathsf{k}[x_1, \ldots, x_n]_f$.

5. a) Prove that the set of $n \times m$ matrices of rank $\leq r$ is an affine algebraic variety, and it contains an open algebraic subvariety formed by all the matrices of rank exactly r.
b) Produce a structure of an algebraic variety on the Grassmannian $\mathrm{Gr}(r, n)$ of r-dimensional linear subspaces in an n-dimensional vector space.

Exercises in Algebraic Geometry 14.10.2021

1. Prove that an A-module M is noetherian iff
a) Any increasing chain of submodules $M_1 \subset M_2 \subset \ldots$ eventually stabilizes: $M_n = M_{n+1}$ for $n \gg 0$.
b) In a short exact sequence $0 \to M' \to M \to M'' \to 0$ both M' and M'' are noetherian.

2. Prove that a) the polynomial algebra in infinitely many variables $\mathsf{k}[x_1, x_2, \ldots]$ is *not* noetherian.
b) $\mathcal{A} = \mathsf{k} \oplus x\mathsf{k}[x, y] \subset \mathsf{k}[x, y]$ is *not* noetherian.

3. Prove that a) a topological space X is irreducible iff every non-empty open subset $U \subset X$ is dense in X.
b) A subset $Z \subset X$ (with induced topology) is irreducible iff its closure $\overline{Z} \subset X$ is irreducible.
c) The image of an irreducible subset $Z \subset X$ under a continuous map $X \to Y$ is irreducible.

4. Let X be a noetherian topological space. Prove that
a) For any open subset $U \subset X$ we have $\dim U \leq \dim X$.
b) If X is a union of open subsets U_i, then $\dim X = \max_i \dim U_i$.
c) If X is irreducible, and $Z \subsetneqq X$ a closed subset, then $\dim X > \dim Z$.
d) If X is an algebraic variety, and X is a union of finitely many locally closed subsets X_i, then $\dim X = \max_i \dim X_i$.
e) Give an example of a noetherian space X such that d) above fails.

5. Let $X = \mathrm{Spec}(\mathcal{A})$ be an irreducible affine algebraic variety. We denote by $\mathsf{k}(X) = \mathrm{Frac}\,\mathcal{A}$ the field of rational functions on X. Prove that the dimension of X equals the transcendence degree of $\mathsf{k}(X)$ over k.

Exercises in Algebraic Geometry 21.10.2021

1. Prove that the open embedding $\mathbb{A}^2 \setminus \{0\} \hookrightarrow \mathbb{A}^2$ is not affine.

2. Prove that a) $\dim(X \times Y) = \dim X + \dim Y$ for algebraic varieties X, Y.
b) $X \times Y$ is irreducible if both X and Y are.

3. a) Find all the irreducible components of the affine algebraic variety $X \subset \mathbb{A}^3$ cut out by the equations $y^2 = xz$ and $z^2 = y^3$.
b) Prove that the field of rational functions on any irreducible component of X is isomorphic to $\mathsf{k}(t)$.

4. Consider the curve $C \subset \mathbb{A}^2$ cut out by the equation $y^2 = x^2 + x^3$.
a) List all the points $c \in C$ where the rational function $t = y/x$ is regular.
b) Prove that $t \notin \mathsf{k}[C]$.

5. Consider the closed subvariety $Z \subset \mathbb{P}^3$ cut out by three homogeneous equations

$$x_1x_3 = x_2^2,\ x_0x_2 = x_1^2,\ x_0x_3 = x_1x_2.$$

a) Prove that Z is irreducible and isomorphic to $\mathbb{P}^1$.
b) Find the irreducible components of the projective variety Z' cut out by two out of three equations above.

Exercises in Algebraic Geometry 28.10.2021

1. Prove that the image of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^\ell \subset \mathbb{P}^{n\ell+n+\ell}$ is not contained in any linear subspace $\mathbb{P}^k$ for $k < n\ell + n + \ell$.

2. For a positive integer ℓ, consider the *Veronese embedding* $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$, where $N = \binom{n+\ell}{\ell} - 1$: for an $n+1$-dimensional k-vector space V it embeds $\mathbb{P}^n = \mathbb{P}(V)$ into $\mathbb{P}(\mathrm{Sym}^\ell V)$, a line $L \subset V$ goes to $\mathrm{Sym}^\ell L \subset \mathrm{Sym}^\ell V$. The monomial coordinates in $\mathbb{P}^N$ are numbered by sequences $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ such that $i_0 + \ldots + i_n = \ell$. In the homogeneous coordinates on $\mathbb{P}^n$, a point $(x_0, \ldots, x_n)$ goes to $(v_{i_0 \ldots i_n} = x_0^{i_0} \cdots x_n^{i_n})$. Prove that

a) it is indeed an embedding, and the image is cut out by quadratic equations $v_{i_0\dots i_n}v_{j_0\dots j_n} = v_{h_0\dots h_n}v_{k_0\dots k_n}$ for all quadruples of sequences such that $i_0+j_0 = h_0+k_0, \dots, i_n+j_n = h_n+k_n$.

b) The image of the Veronese embedding is not contained in any linear subspace $\mathbb{P}^m$ for $m < N$.

c) The projective variety $Z \subset \mathbb{P}^3$ of problem 5 of October 21 is the image of the Veronese embedding of $\mathbb{P}^1$.

3. a) Let Y be a separable algebraic variety, and $\varphi, \psi\colon X \to Y$ be two morphisms from an algebraic variety X that coincide on an open dense subvariety $U \subset X$. Prove that $\varphi = \psi$.

b) If we have a morphism $X \supset U \xrightarrow{\pi} Y$, where Y is separable, and $U \subset X$ is open dense, prove that there is a maximal open subvariety $U \subset \widetilde{U} \subset X$ such that π extends to a morphism $\widetilde{U} \to Y$.

4. Consider a *rational* morphism $\psi\colon \mathbb{P}^2 \to \mathbb{P}^2$ (i.e. a morphism $\mathbb{P}^2 \supset U \to \mathbb{P}^2$ defined on a dense open subvariety) given in the homogeneous coordinates by the formula $y_0 = x_1x_2,\ y_1 = x_0x_2,\ y_2 = x_0x_1$.

a) Find the maximal open subvariety $\widetilde{U} \subset \mathbb{P}^2$ such that ψ is a morphism from $\widetilde{U}$ to $\mathbb{P}^2$ (domain of regularity of ψ).

b) Prove that ψ^{-1} is also rational (i.e. ψ is *birational*) and find the maximal open $\widehat{U} \subset \mathbb{P}^2$ such that ψ^{-1} is a morphism from $\widehat{U}$ to $\mathbb{P}^2$.

c) What are the maximal open subsets $U_1 \subset \mathbb{P}^2$ and $U_2 \subset \mathbb{P}^2$ such that ψ and ψ^{-1} define the mutually inverse isomorphisms of U_1 and U_2?

5. Let $X \subset \mathbb{P}^3$ be an irreducible surface cut out by a quadratic equation.

a) Construct a birational isomorphism $\psi\colon X \dashrightarrow \mathbb{P}^2$ (*stereographic projection*).

b) Find the domain or regularity of ψ.

c) Find the domain of regularity of ψ^{-1}.

Exercises in Algebraic Geometry 04.11.2021

1. For a point $x \in X$ of an algebraic variety we define the local dimension $\dim_x X$ as the minimum of $\dim U$ for all open neighbourhoods U of x.

a) Prove that for any point x of an irreducible algebraic variety X, the local dimension $\dim_x X$ equals $\dim X$.

b) Give an example of an irreducible noetherian space X such that a) above fails.

2. Let $\pi\colon X \to Y$ be a morphism of algebraic varieties. Prove that

a) if π is *dominant* (i.e. its image is dense in Y), then $\dim X \geq \dim Y$.

b) If π has finite fibers, then $\dim X \leq \dim Y$.

c) If π has finite fibers, then their cardinalities are bounded by some constant.

3. Let X be an affine algebraic variety. Fix a finite dimensional subspace $V \subset \mathsf{k}[X]$, and for $d \in \mathbb{N}$ set $D := V^{\oplus d}$. Every point $\underline{v} = (v_1, \dots, v_d) \in D$ defines a homomorphism of

algebras $\mathsf{k}[x_1, \dots, x_d] \to \mathsf{k}[X]$, $x_i \mapsto v_i$, and hence a morphism $\pi_{\underline{v}} \colon X \to \mathbb{A}^d$. Consider the subset $U \subset D$ formed by all the points $\underline{v}$ such that $\pi_{\underline{v}}$ is finite.

a) Prove that U is open. In particular, it is dense if not empty.

b) We define $\dim' X$ as the minimal $d \in \mathbb{N}$ such that there is a finite morphism $X \to \mathbb{A}^d$. Prove that for an open cover $X = \bigcup U_i$ we have $\dim' X = \max_i \dim' U_i$.

c) Prove that for a finite surjective morphism $\pi \colon X \to Y$ we have $\dim' X = \dim' Y$.

d) Prove that $\dim' \mathbb{A}^d = d$.

4. Compute the dimension of the

a) algebraic variety formed by all the $n \times m$ matrices of rank r.

b) Grassmannian $\mathrm{Gr}(r, n)$ of r-dimensional linear subspaces in an n-dimensional vector space.

c) Algebraic variety whose points are all quadratic hypersurfaces in $\mathbb{P}^6$.

5. Let $X \subset \mathbb{P}^6$ be a closed 2-dimensional subvariety (a projective surface). Consider the set L_X of all lines $\mathbb{P}^1 \subset \mathbb{P}^6$ having nonempty intersection with X.

a) Prove that L_X is an algebraic variety.

b) Compute $\dim L_X$.

c) Let $X, Y, Z \subset \mathbb{P}^6$ be three projective surfaces. Prove that if there exists a line $\mathbb{P}^1 \subset \mathbb{P}^6$ intersecting all the three surfaces, then there are infinitely many such lines.

Exercises in Algebraic Geometry 11.11.2021

1. Let $\pi \colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ (resp. $\mathcal{G}$) be a sheaf on X (resp. on Y).

a) For an open set $V \subset Y$ set $\pi_*\mathcal{F}(V) := \mathcal{F}(\pi^{-1}(V))$. Prove that $\pi_*\mathcal{F}$ is a sheaf on Y (*direct image* or *pushforward*).

b) For on open set $U \subset X$ consider $\lim\limits_{V \supset \pi(U)} \mathcal{G}(V)$. Give an example where this is not a sheaf (just a presheaf). Its sheafification is denoted $\pi^{-1}\mathcal{G}$ (*inverse image* or *pullback*).

c) Construct natural morphisms $\pi^{-1}\pi_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to \pi_*\pi^{-1}\mathcal{G}$.

d) Prove that $\mathrm{Hom}(\pi^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{F})$ (i.e. the functor π^{-1} is *left adjoint* to the functor π_*, while π_* is *right adjoint* to π^{-1}).

2. Let $x \in X$, and let A be an abelian group. We define the *skyscraper sheaf* A_x as follows: $A_x(U) = 0$ if $x \notin U$, and $A_x(U) = A$ if $x \in U$. Prove that $A_x = i_*\underline{A}$, where $i \colon \overline{x} \hookrightarrow X$ is the embedding of the closure $\overline{x}$ of x into X, and $\underline{A}$ is the constant sheaf on $\overline{x}$.

3. Let $j \colon U \hookrightarrow X$ be the embedding of an open subset U, and let $i \colon Z \hookrightarrow X$ be the embedding of its closed complement Z. We consider the sheaves of abelian groups.

a) Prove that for a sheaf $\mathcal{G}$ on Z, the stalk $(i_*\mathcal{G})_x$ equals $\mathcal{G}_x$ if $x \in Z$, and $(i_*\mathcal{G})_x = 0$ if $x \notin Z$.

b) For a sheaf $\mathcal{H}$ on U we define $j_!\mathcal{H}$ (*extension by zero* or *shriek extension*) as the sheafification of the following presheaf on $X\colon V \mapsto \mathcal{H}(V)$ if $V \subset U$, and $V \mapsto 0$ if $V \not\subset U$. Prove that the stalk $(j_!\mathcal{H})_x = \mathcal{H}_x$ if $x \in U$, and $(j_!\mathcal{H})_x = 0$ if $x \notin U$.

c) Prove that we have a short exact sequence of sheaves $0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$ for a sheaf $\mathcal{F}$ on X .

4. For a section $\sigma \in \mathcal{F}(V)$ of a sheaf $\mathcal{F}$ of abelian groups on X we define its support $\mathrm{supp}(\sigma) \subset V$ as the set of points $x \in V$ such that the image σ_x of σ in the stalk $\mathcal{F}_x$ does not vanish.

a) Prove that $\mathrm{supp}(\sigma)$ is a closed subset of V.

b) For a closed subset $Z \subset X$ we define $\mathcal{F}_Z(V)$ as a subgroup in $\mathcal{F}(V)$ formed by all the sections with support in $Z \cap V$. Prove that $V \mapsto \mathcal{F}_Z(V)$ is a sheaf. It is called the subsheaf with supports in Z.

c) Prove that we have a short exact sequence of sheaves $0 \to \mathcal{F}_Z \to \mathcal{F} \to j_*j^{-1}\mathcal{F}$ in notations of Problem 3.

d) Prove that if $\mathcal{F}$ is flabby, then this s.e.s. extends to $0 \to \mathcal{F}_Z \to \mathcal{F} \to j_*j^{-1}\mathcal{F} \to 0$.

5. Let $\mathcal{O}_X$ be the structure sheaf of an algebraic variety X, and let $Z \subset X$ be a closed subvariety. For an open $U \subset X$ we denote by $\mathcal{I}_Z(U) \subset \mathcal{O}_X(U)$ the ideal of regular functions vanishing at $Z \cap U$. Prove that

a) $U \mapsto \mathcal{I}_Z(U)$ is a sheaf (*sheaf of ideals of* Z).

b) The quotient sheaf $\mathcal{O}_X/\mathcal{I}_Z$ is isomorphic to $i_*\mathcal{O}_Z$, where $i\colon Z \to X$ is the closed embedding, and $\mathcal{O}_Z$ is the structure sheaf of Z.

Exercises in Algebraic Geometry 18.11.2021

1. a) Let X be an algebraic variety (not necessarily an affine one), and let $f \in \mathsf{k}[X] = A$, and let $X_f \subset X$ be the open subset formed by all the points $x \in X$ such that $f(x) \neq 0$. Prove that $\mathsf{k}[X_f] = A_f$ (localization).

b) Prove that X is affine iff there is a finite set $f_1, \ldots, f_n \in A$ such that X_{f_i} are all affine, and $f_1, \ldots, f_n$ generate the unit ideal $(1) = A$.

2. Prove that a morphism $\pi\colon X \to Y$ is affine iff for *any* affine open $V \subset Y$ its preimage $\pi^{-1}(V)$ is affine as well.

3. a) Let $\mathcal{A}$ be a quasicoherent sheaf of $\mathcal{O}_Y$-algebras (i.e. $\mathcal{A}(V)$ is a finitely generated commutative $\mathsf{k}[V]$-algebra without nilpotents for any open $V \subset Y$). Prove that there is a unique algebraic variety X and a morphism $\pi\colon X \to Y$ such that for any affine open $V \subset Y$ we have an isomorphism $\pi^{-1}(V) \simeq \mathrm{Spec}(\mathcal{A}(V))$, and for any affine open embedding $U \subset V$ the morphism $\pi^{-1}(U) \hookrightarrow \pi^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. This variety X is denoted $\mathrm{Spec}(\mathcal{A})$.

b) Prove that the morphism $\pi\colon X = \mathrm{Spec}(\mathcal{A}) \to Y$ is affine, and $\mathcal{A} = \pi_*\mathcal{O}_X$. Conversely, if $\pi\colon X \to Y$ is an affine morphism, then $\mathcal{A} := \pi_*\mathcal{O}_X$ is a quasicoherent sheaf of $\mathcal{O}_Y$-algebras, and $X = \mathrm{Spec}(\mathcal{A})$.

c) Prove that π_* gives rise to an equivalence of categories of quasicoherent $\mathcal{O}_X$-modules and the category of $\mathcal{A}$-modules (i.e. quasicoherent $\mathcal{O}_Y$-modules equipped with a structure of $\mathcal{A}$-module).

4. A *vector bundle* of rank n over an algebraic variety Y is a morphism of algebraic varieties $\pi\colon \mathcal{V} \to Y$ with the following auxiliary structure: for some open cover $Y = \bigcup U_i$ we have isomorphisms $\psi_i\colon \pi^{-1}(U_i) \to \mathbb{A}^n \times U_i$ such that for any i, j and any open affine subvariety $W = \mathrm{Spec}(A) \subset U_i \cap U_j$, the automorphism $\psi_{ij} := \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n \times W = \mathrm{Spec}(A[t_1, \dots, t_n])$ is induced by an A-linear automorphism ϕ of the algebra $A[t_1, \dots, t_n]$, i.e. $\phi(a) = a$ for any $a \in A$, and $\phi(t_k) = \sum a_{kl} t_l$, where $a_{kl} \in A$. An *isomorphism* $\eta\colon (\mathcal{V}, \pi, \{U_i\}, \{\psi_i\}) \to (\mathcal{V}', \pi', \{U'_k\}, \{\psi'_k\})$ of vector bundles is an isomorphism of the algebraic varieties $\eta\colon \mathcal{V} \to \mathcal{V}'$ such that $\pi = \pi' \circ \eta$, and $\mathcal{V}, \pi$ along with the cover $Y = \bigcup U_i \cup \bigcup U'_k$ and isomorphisms $\psi_i, \psi'_k \circ \eta$ also define a structure of vector bundle on $\mathcal{V}$. Finally, let $\mathcal{E}$ be a locally free sheaf of rank n on Y generating the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_Y}\mathcal{E}$, and $\mathcal{V} := \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_Y}\mathcal{E})$ with the projection morphism $\pi\colon X \to Y$. For any open affine $U \subset Y$ such that the restriction $\mathcal{E}|_U$ is free we choose a basis of sections in $\mathcal{E}|_U$, and let $\psi\colon \pi^{-1}(U) \to \mathbb{A}^n \times U$ be the isomorphism we get under identification of $\mathrm{Sym}_{\mathcal{O}_Y(U)}\mathcal{E}(U)$ with $\mathcal{O}_Y(U)[t_1, \dots, t_n]$.

a) Prove that $(\mathcal{V}, \pi, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y that does not depend on the choice of bases in $\mathcal{E}|_U$ up to isomorphism. We will call it the vector bundle *associated with* $\mathcal{E}$, and we will denote it $\mathrm{Vect}(\mathcal{E})$.

b) A *section* of a morphism $\pi\colon X \to Y$ over an open $U \subset Y$ is a morphism $\sigma\colon U \to X$ such that $\pi \circ \sigma = \mathrm{Id}_U$. Clearly, the sections form a sheaf of sets to be denoted $\mathrm{Sect}_{X/Y}$. Prove that if $\pi\colon \mathcal{V} \to Y$ is a vector bundle of rank n, then the sheaf $\mathrm{Sect}_{\mathcal{V}/Y}$ has a natural structure of $\mathcal{O}_Y$-module, and it is a locally free $\mathcal{O}_Y$-module of rank n.

c) Let $\mathcal{E}$ be a locally free sheaf of rank n on Y, and $\mathcal{V} = \mathrm{Vect}(\mathcal{E})$, and let $\mathcal{S} = \mathrm{Sect}_{\mathcal{V}/Y}$ be the sheaf of sections of $\mathcal{V}$ over Y. Prove that $\mathcal{S} \simeq \mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)$, and this isomorphism can be constructed as follows. A section $\sigma \in \mathcal{E}^\vee(U)$ over an open $U \subset Y$ can be viewed as an element $\sigma \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \mathcal{O}_U)$, so that σ defines a homomorphism of $\mathcal{O}_U$-algebras $\mathrm{Sym}_{\mathcal{O}_U}\mathcal{E}|_U \to \mathcal{O}_U$, that in turn defines a morphism of spectra $U = \mathrm{Spec}(\mathcal{O}_U) \to \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_U}\mathcal{E}|_U) = \pi^{-1}(U)$, i.e. a section of $\mathcal{V}$ over U. So you have to check that this assignment gives the desired isomorphism from $\mathcal{E}^\vee$ onto $\mathcal{S}$.

d) Prove that the above construction defines a bijection between the isomorphism classes of locally free sheaves of rank n on Y and the isomorphism classes of vector bundles of rank n over Y.

5. Let $V = \mathsf{k}^{n+1}$, and $\mathbb{P}^n = \mathbb{P}(V)$. In the trivial vector bundle $\mathbb{P}^n \times V$ we consider the tautological vector subbundle $\mathcal{S}$ of rank 1 (*line bundle*) (its fiber over $x \in \mathbb{P}^n$ is the corresponding line $\ell_x \subset V$). The corresponding rank 1 locally free sheaf of sections (*invertible sheaf*) is denoted $\mathcal{O}_{\mathbb{P}^n}(-1)$. Its dual $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n})$ is denoted $\mathcal{O}_{\mathbb{P}^n}(1)$. For $n \in \mathbb{N}$ we set $\mathcal{O}_{\mathbb{P}^n}(\pm n) := \mathcal{O}_{\mathbb{P}^n}(\pm 1)^{\otimes n}$ (tensor product over $\mathcal{O}_{\mathbb{P}^n}$). Prove that

a) The global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n)) = 0$.

b) The global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)) = \mathrm{Sym}^n V^*$.

Exercises in Algebraic Geometry 25.11.2021

We know already that a hypersurface $H \subset \mathbb{P}^N$ is the zero set of a homogeneous polynomial in $N+1$ variables. Geometrically, the degree d of this polynomial is the number of points (with multiplicities) of the intersection $H \cap \ell$ with a generic line $\ell \subset \mathbb{P}^N$. An *effective divisor* D on $\mathbb{P}^N$ of degree d is a formal linear combination $D = \sum c_i H_i$ of hypersurfaces with coefficients $c_i \in \mathbb{N}$ such that $\sum c_i d_i = d$ ($d_i = \deg H_i$). The set of degree d effective divisors is naturally isomorphic to the set of degree d polynomials in $N+1$ variables up to proportionality, i.e. to $\mathbb{P}\mathrm{Sym}^d(\mathsf{k}^{N+1})$. In other words, degree d effective divisors are naturally parametrized by the algebraic variety $\mathbb{P}\mathrm{Sym}^d(\mathsf{k}^{N+1})$; moreover, an open subvariety parametrizes the irreducible divisors. In this assignment we will arrive at a similar parametrization for effective *algebraic cycles* (formal positive linear combinations of irreducible subvarieties in $\mathbb{P}^N$ of dimension $n < N$): *Chow coordinates.*

1. Let $\pi\colon X \to Y$ be a surjective morphism of irreducible algebraic varieties of dimensions n, m. Prove that

a) For any $y \in Y$ and any irreducible component Z of the fiber $\pi^{-1}(y)$ we have $\dim Z \geq n - m$.

b) There is an open $U \subset Y$ such that for any $y \in U$ we have $\dim \pi^{-1}(y) = n - m$.

c) $\dim \pi^{-1}(y)$ is an upper-semicontinuous function on Y (i.e. for any $k \in \mathbb{N}$, the set $\{y \in Y : \dim \pi^{-1}(y) \leq k\}$ is open in Y) if π is additionally assumed to be proper.

d) Give an example where $\dim \pi^{-1}(y)$ is *not* upper-semicontinuous when π is not assumed to be proper.

e) If both X and Y are projective, and all the fibers $\pi^{-1}(y)$ are irreducible of the same dimension (but we do not assume anymore that X is irreducible, though we keep the assumption that Y is irreducible), then X is irreducible.

2. The set of hyperplanes in $\mathbb{P}^N$ is naturally parametrized by the points of the dual projective space $\mathbb{P}^\vee$ (nonzero linear functionals on k^{N+1} up to proportionality). Let $X \subset \mathbb{P}^N$ be an irreducible projective subvariety of dimension n. We consider the closed subvariety $\Gamma \subset (\mathbb{P}^\vee)^{n+1} \times X$ formed by all the collections $(\xi_0, \ldots, \xi_n, x)$ such that all the hyperplanes $\xi_i \subset \mathbb{P}^N$ contain the point $x \in X$. We have two projections: $\phi\colon \Gamma \to (\mathbb{P}^\vee)^{n+1}$, $\psi\colon \Gamma \to X$. Prove that

a) Γ is irreducible of dimension $N(n+1) - 1$.

b) $\phi(\Gamma)$ is an irreducible closed subvariety of $(\mathbb{P}^\vee)^{n+1}$ of codimension 1.

c) There is a polynomial F_X in $n+1$ groups of $N+1$ variables, homogeneous of degree d_i in each group, with all factors of multiplicity 1, such that $\phi(\Gamma)$ is cut out by a single equation F_X. It is called the *associated form* of the subvariety X, and its coefficients are called the *Chow coordinates* of X.

3. Prove that a point $x \in \mathbb{P}^N$ lies in X iff any $n+1$ hyperplanes $\xi_0, \ldots, \xi_n$ containing x satisfy the equation $F_X(\xi_0, \ldots, \xi_n) = 0$. Hence X can be uniquely reconstructed from F_X.

4. Prove that a) all the degrees d_i of Problem 2 are equal. We will denote them by d.

b) d is the maximal cardinality of the intersection $E \cap X$, where $E \subset \mathbb{P}^N$ is a linear subspace of dimension $N-n$ such that $E \cap X$ is finite. This number is called the *degree* of X.

The set of all nonzero forms F in $n+1$ groups of $N+1$ variables, homogeneous of degree d in any group, up to proportionality, is the set of points of an appropriate projective space $\mathbb{P}^{M(N,n,d)}$. One can check that the forms F equal to F_X for some irreducible projective subvariety X of dimension n and degree d form a quasiprojective subvariety $C_{N,n,d} \subset \mathbb{P}^{M(N,n,d)}$. If we associate to a cycle $Y = \sum n_i X_i$ ($\dim X_i = n$, $\sum n_i \deg(X_i) = d$, and all the subvarieties X_i are irreducible) the form $F_Y := \prod F_{X_i}^{n_i}$, then the forms F equal to F_Y for some n-dimensional cycle Y of degree d form a *projective* subvariety $B_{N,n,d}$, containing $C_{N,n,d}$ as an open (but *not necessarily* dense) subset. It is called the *Chow variety.*

5. Prove that $B_{3,1,3}$ has 4 irreducible components of dimension 12.

Exercises in Algebraic Geometry 02.12.2021

The goal of this assignment is an algebraic definition of the degree of a projective variety, that is the *Hilbert polynomial.*

1. A polynomial $P \in \mathbb{Q}[z]$ is called *integral* if $P(n) \in \mathbb{Z}$ for $n \gg 0$.

a) Prove that there are integers $c_0, \ldots, c_r$ such that $P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \ldots + c_r$. In particular, $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

b) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be an arbitrary function. Assume that there is an integral polynomial $Q(z)$ such that the finite difference function $\Delta f(n) = f(n+1) - f(n)$ coincides with $Q(n)$ for $n \gg 0$. Prove that there is an integral polynomial $P(z)$ such that $f(n) = P(n)$ for all $n \gg 0$.

2. Let $M = \oplus M_d$ be a finitely generated graded module over $A = \mathsf{k}[t_0, \ldots, t_n]$. For $l \in \mathbb{Z}$ we define the *twisted* module $M(l)$ as $M(l)_d := M_{d+l}$. We define a homogeneous ideal $\mathrm{Ann}(M) \subset A$ (the annihilator of M) as $\{s \in A \ : \ s \cdot M = 0\}$. Prove that

a) There is a filtration $0 = M^0 \subset M^1 \subset \ldots \subset M^r = M$ such that for any i we have $M^i/M^{i-1} \simeq (A/\mathfrak{p}_i)(l_i)$, where $\mathfrak{p}_i \subset A$ is a homogeneous prime ideal, and $l_i \in \mathbb{Z}$.

b) If $\mathfrak{p}$ is a homogeneous prime ideal in A, then $\mathfrak{p} \supset \mathrm{Ann}(M)$ iff $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. In particular, the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are exactly the minimal homogeneous prime ideals containing $\mathrm{Ann}(M)$.

c) For any such ideal $\mathfrak{p}$ the number of its occurences in the (multi)set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ equals the length of the localized module $M_\mathfrak{p}$ over the local ring $A_\mathfrak{p}$. In particular, this number is independent of the choice of a filtration in a) above. It is called the *multiplicity* of M at $\mathfrak{p}$ and denoted $\mu_\mathfrak{p}(M)$.

3. We define the Hilbert function ϕ_M of a finitely generated graded A-module M as $\phi_M(l) := \dim M_l$. Prove that

a) There is a unique integral polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\phi_M(l) = P_M(l)$ for $l \gg 0$ (the *Hilbert polynomial*).

b) $\deg P_M$ is equal to the dimension of the projective variety in $\mathbb{P}^n$ defined by the ideal $\mathrm{Ann}(M)$.

For a projective variety $Y \subset \mathbb{P}^n$ its Hilbert polynomial P_Y is defined as the Hilbert polynomial of the A-module $\mathsf{k}[C(Y)]$ (the homogeneous coordinate ring of Y). If $\dim Y = r$, the leading coefficient of P_Y multiplied by $r!$ is called the *degree* of Y.

4. Prove that a) $\deg Y$ is a positive integer.

b) Let $Y = Y_1 \cup Y_2$, where $\dim Y_1 = \dim Y_2 = r > \dim(Y_1 \cap Y_2)$. Then $\deg Y = \deg Y_1 + \deg Y_2$.

c) $\deg \mathbb{P}^n = 1$.

d) If $H \subset \mathbb{P}^n$ is a hypersurface cut out by a homogeneous polynomial of degree d, then $\deg H = d$.

e) If $Y \not\subset H$, and Y is equidimensional of dimension r, then we know that $Y \cap H = Z_1 \cup \ldots \cup Z_s$, where Z_j are irreducible subvarieties of dimension $r-1$. The *multiplicity of intersection* of Y and H along Z_j is defined as $\iota(Y, H; Z_j) := \mu_{\mathfrak{p}_j}(A/(I_Y + I_H))$, where $\mathfrak{p}_j, I_Y, I_H$ are the homogeneous ideals of the varieties Z_j, Y, H. Prove that

$$\sum_{j=1}^{s} \iota(Y, H; Z_j) \cdot \deg Z_j = \deg Y \cdot \deg H$$

(*Bezout theorem*).

f) $\deg Y$ equals the number of points of intersection of Y with a general linear subspace of complementary dimension. Thus the definitions of this home assignment and the previous one match.

5. Prove that a) the degree of the image of the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^{(r+1)(s+1)-1}$ equals $\binom{r+s}{r}$.

b) The degree of the image of the Veronese embedding of $\mathbb{P}^r$ into $\mathbb{P}(\mathrm{Sym}^d \mathsf{k}^{r+1})$ equals d^r.

Exercises in Algebraic Geometry 09.12.2021

1. a) Let $C \subset \mathbb{A}^3$ be a reducible algebraic curve equal to the union of 3 coordinate lines. Prove that the ideal $J_C \subset \mathsf{k}[x, y, z]$ can not be generated by 2 elements.

b) Give an example of an affine curve that can not be a closed subvariety in $\mathbb{A}^{2021}$.

c) Give an example of an irreducible affine curve that can not be a closed subvariety in $\mathbb{A}^{2021}$.

2. a) Let $X \subset \mathbb{P}^2$ be a reducible 0-dimensional subvariety: a union of 3 points not lying on a line $\mathbb{P}^1 \subset \mathbb{P}^2$. Prove that the ideal of X in the homogeneous coordinate ring of $\mathbb{P}^2$ can not be generated by 2 elements.

b) Let $Y \subset \mathbb{A}^2$ be a union of finitely many points. Prove that Y can be cut out by 2 equations. Hence the ideal $J_Y \subset \mathsf{k}[x, y]$ is the radical of an ideal generated by 2 elements.

c) Let $Z \subset \mathbb{P}^2$ be a union of finitely many points. Prove that Z can be cut out by 2 homogeneous equations.

3. Let $C \subset \mathbb{A}^3$ be a closed irreducible algebraic curve not equal to a vertical line.

a) Prove that there is a polynomial $f(x, y)$ such that $f|_C \equiv 0$ (here x, y, z are the coordinates on $\mathbb{A}^3$).

b) Prove that all such polynomials form a principal ideal $I = (P) \subset \mathsf{k}[x, y]$.

c) Prove that the curve cut out by P in $\mathbb{A}^2$ is the closure of the image of $C \subset \mathbb{A}^3$ under the projection $\mathbb{A}^3 \to \mathbb{A}^2,\ (x, y, z) \mapsto (x, y)$.

4. Let $h(x, y, z) = h_0(x, y)z^n + \ldots + h_n(x, y) \in J_C$ be a polynomial of the minimal positive degree in z vanishing on $C \subset \mathbb{A}^3$ and such that $h_0(x, y)$ is not divisible by P.

a) Prove that for $f(x, y, z) = f_0(x, y)z^m + \ldots + f_m(x, y) \in J_C$ we have $f \cdot h_0^m = h \cdot g + Q$, where Q is divisible by P.

b) Prove that the equations $h = 0 = P$ cut out a reducible curve $C' \subset \mathbb{A}^3$ consisting of C and a few (possibly none) vertical lines defined by equations $h_0 = 0 = P$.

5. Prove that a) any closed curve $C \subset \mathbb{A}^3$ can be cut out by 3 equations.

b) Any closed curve $X \subset \mathbb{P}^3$ can be cut out by 3 homogeneous equations.

Exercises in Algebraic Geometry 16.12.2021

1. Let $\mathcal{F}$ be a coherent sheaf on an open subvariety $U \subset X$ of an algebraic variety. Prove that there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U = \mathcal{F}$.

2. a) Prove that the Grassmannian $\mathrm{Gr}(2, 4)$ can be embedded into $\mathbb{P}^5$ as a quadratic hypersurface Π (*Plücker quadric*). Thus the points of the Plücker quadric parametrize the lines $\mathbb{P}^1 \subset \mathbb{P}^3$.

b) Prove that any plane $\mathbb{P}^2 \subset \Pi \subset \mathbb{P}^5$ is of the following kind: it is either the set of lines $\ell \subset \mathbb{P}^3$ containing a point $p \in \mathbb{P}^3$, or the set of lines $\ell \subset \mathbb{P}^3$ contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$.

3. Let $X \subset \mathbb{P}^3$ be a surface of degree d cut out by a homogeneous polynomial F (thus X is represented by a point of $\mathbb{P}^N,\ N = \frac{1}{6}(d + 1)(d + 2)(d + 3) - 1$, given by F up to proportionality). Prove that

a) the incidence subset $\Gamma = \{(\ell, F)\ :\ \ell \subset X\} \subset \Pi \times \mathbb{P}^N$ is actually a closed subvariety (cut out by equations homogeneous in coefficients of F and in Plücker coordinates on Π).

b) Γ is irreducible.

c) $\dim \Gamma = \frac{1}{6}d(d + 1)(d + 5) + 3$.

d) If $d > 3$, then a surface X corresponding to a point of an open (nonempty) subset $U \subset \mathbb{P}^N$ contains no lines.

4. Prove that a) the cubic surface $X_0 \subset \mathbb{P}^3$ cut out by the equation $x_1x_2x_3 = x_0^3$ contains exactly three lines.

b) Any cubic surface $X \subset \mathbb{P}^3$ contains at least one line.

c) There is a nonempty open subset $U \subset \mathbb{P}^{19}$ parametrizing cubic surfaces such that any cubic surface corresponding to a point of U contains finitely many lines.

d) There are cubic surfaces containing infinitely many lines.

5. a) Let $X \subset \mathbb{P}^3$ be a smooth quadric. Prove that the subset $\Pi_X \subset \Pi$ formed by all the lines lying in X is the union of 2 nonintersecting curves, each of which is isomorphic to $\mathbb{P}^1$.

b) Prove that the set of non-smooth quadrics forms a hypersurface in $\mathbb{P}^9$.

Exercises in Algebraic Geometry 18.01.2022

1. Let $f \in \mathsf{k}[x_1, \dots, x_n]$. Prove $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

2. Let X and Y be affine algebraic varieties. Prove

$$\Omega[X \times Y] = \Omega[X] \otimes_{\mathsf{k}[X]} \mathsf{k}[X \times Y] \oplus \Omega[Y] \otimes_{\mathsf{k}[Y]} \mathsf{k}[X \times Y].$$

3. Let $C \subset \mathbb{A}^2$ be the affine algebraic curve cut out by $x^2 + y^2 = 1$. Let $\operatorname{char} \mathsf{k} \neq 2$. Prove that

a) Any regular function on C can be uniquely written in the form $f(y) + xg(y)$ for some polynomials f, g.

b) Any regular differential form on C can be uniquely written in the form $f(y)dx + (g(y) + xh)dy$ for some polynomials f, g and a constant $h \in \mathsf{k}$.

c) The differential forms $\frac{dx}{y}$ on D_y and $-\frac{dy}{x}$ on D_x coincide on the intersection D_{xy} of these open sets. Thus they give rise to a global regular differential form ω on C.

d) Write down ω in the form of b).

4. Consider the elliptic curve $E \subset \mathbb{P}^2$ cut out by $y^2 z = x(x - z)(x - \lambda z)$, where $\lambda \in \mathsf{k}$ is a fixed parameter distinct from 0 and 1. In the open chart $D_z \cap E$, consider the differential form $\omega = \frac{dx}{y}$. Prove that

a) ω is regular on $D_z \cap E$.

b) ω extends (uniquely) to a regular differential form on the whole of E.

c) ω vanishes nowhere on E.

d) Any global regular differential form on E is a scalar multiple of ω.

5. Prove that any algebraic group is smooth at any point.

Exercises in Algebraic Geometry 25.01.2022

1. Recall the correspondence between locally free sheaves and vector bundles of Problem 4 of 18.11.2021. For a smooth algebraic variety X let T^*X denote the cotangent vector bundle over X whose local sections form the sheaf of Kähler differentials Ω_X. Let $X = \mathrm{Gr}(r, n)$ be the Grassmannian of r-dimensional linear subspaces in an n-dimensional vector space V. It is equipped with two vector bundles $\mathcal{S}, \mathcal{Q}$: the fiber of $\mathcal{S}$ over an r-dimensional subspace $U \subset V$ is U, and the fiber of $\mathcal{Q}$ over U is V/U. Prove that $T^*\mathrm{Gr}(r, n) \cong \mathcal{Q}^* \otimes \mathcal{S}$ (where $\mathcal{Q}^*$ stands for the dual vector bundle to $\mathcal{Q}$).

2. Assume the characteristic of k is not 2. Let V be a $2n$-dimensional vector space equipped with a nondegenerate bilinear skew-symmetric, i.e. *symplectic* form $\langle\, ,\rangle$. An n-dimensional subspace $L \subset V$ is called *Lagrangian* if $\langle\, ,\rangle|_L = 0$. Prove that

a) The set of all Lagrangian subspaces $L \subset V$ forms a closed subvariety of $\mathrm{Gr}(n, 2n)$, denoted $\mathrm{LGr}(V)$: *Lagrangian Grassmannian.*

b) $T^*\mathrm{LGr}(V) \cong \mathrm{Sym}^2\mathcal{S}$.

3. Assume the characteristic of k is not 2. Let V be a $2n$-dimensional vector space equipped with a nondegenerate bilinear symmetric form $(\, ,)$. An n-dimensional subspace $L \subset V$ is called *self-orthogonal* if $(\, ,)|_L = 0$. Prove that

a) The set of all self-orthogonal subspaces $L \subset V$ forms a closed subvariety of $\mathrm{Gr}(n, 2n)$, denoted $\mathrm{SGr}(V)$: *self-orthogonal Grassmannian.*

b) $\mathrm{SGr}(V)$ is disconnected.

c) $T^*\mathrm{SGr}(V) \cong \Lambda^2\mathcal{S}$.

4. Let $F_0(x_0, \ldots, x_n), \ldots, F_n(x_0, \ldots, x_n)$ be homogeneous polynomials of degrees $d_0, \ldots, d_n$. Here $x_0, \ldots, x_n$ are coordinates on an $n+1$-dimensional vector space V. Let $\Gamma \subset \mathbb{P}(V) \times \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$ be the closed subvariety formed by all the collections $(\underline{x}, F_0, \ldots, F_n)$ such that $F_0(\underline{x}) = \ldots = F_n(\underline{x}) = 0$. Let $\varphi\colon \Gamma \to \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$ denote the projection. Prove that

a) $\dim \Gamma + 1 = \dim \varphi(\Gamma) + 1 = \dim \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$.

b) There exists a polynomial $R(F_0, \ldots, F_n)$ in coefficients of F_i such that $R = 0$ iff the system $F_0 = \ldots = F_n = 0$ has a nonzero solution.

5. Let $X \subset \mathbb{P}^n$ be a hypersurface cut out by a homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree d.

a) Prove that the singular points of X are the solutions of the system of equations $F = \frac{\partial F}{\partial x_0} = \ldots = \frac{\partial F}{\partial x_n} = 0$.

b) Prove that if d is not divisible by the characteristic of k, then the first equation $F = 0$ in a) follows from the other ones.

c) Assume k has charactertic 0. Prove that the set S of all $F \in \mathbb{P}(\mathrm{Sym}^d V^*)$ defining singular hypersurfaces in $\mathbb{P}^n$ forms a hypersurface $S \subset \mathbb{P}(\mathrm{Sym}^d V^*)$.

Exercises in Algebraic Geometry 01.02.2022

1. Consider a cubic curve $C_a \subset \mathbb{P}^2$ cut out by the equation

$$x^3 + y^3 + z^3 + a(x+y+z)^3 = 0.$$

a) Find all a such that C_a is singular.

b) Find all the singular points of C_a.

c) For which a the curve C_a is irreducible?

d) Prove that if a cubic curve $C \subset \mathbb{P}^2$ has 3 singular points, then C is a union of 3 lines.

e) Prove that if a cubic curve $C \subset \mathbb{P}^2$ is irreducible and not smooth, then it is birationally isomorphic to $\mathbb{P}^1$.

2. Prove that if a degree $d \geq 2$ hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace $L \simeq \mathbb{P}^r \subset \mathbb{P}^n,\ r \geq n/2$, then X can not be smooth.

3. a) Let $X \subset \mathbb{P}(V)$ be a smooth hypersurface of degree $d \geq 2$. Prove that the set of all hyperplanes $H \subset \mathbb{P}(V)$ tangent to X at some (varying) point $x \in X$ forms a hypersurface $X^\vee \subset \mathbb{P}(V^*)$. In case X is singular, we define $X^\vee$ as the closure of the set of hyperplanes tangent to X at the smooth points of X.

b) Determine the dual curve $X^\vee$ for the curve $X \subset \mathbb{P}^2$ cut out by the equation

$$x^3 + y^3 + z^3 = 0.$$

4. Find all the singular points of the a) *Steiner surface* cut out by the equation

$$x_1^2x_2^2 + x_0^2x_2^2 + x_0^2x_1^2 - x_0x_1x_2x_3 = 0.$$

b) *Dual Steiner surface* cut out by the equation

$$y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3 = 0.$$

5. Assume k has charactertic 0. a) Prove that almost all (that is, all but finitely many) fibers of a function $f\colon \mathbb{A}^n \to \mathbb{A}^1$ are smooth.

b) Prove that for a morphism $\pi\colon X \to Y$ of smooth algebraic varieties, there is a nonempty open subset $U \subset Y$ such that for any $y \in U$ the fiber $\pi^{-1}(y)$ is smooth.

c) Prove the following stronger version of Bertini theorem. Let $X \subset \mathbb{P}^n$ be a smooth irreducible subvariety, and let $L \simeq \mathbb{P}^{n-2} \subset \mathbb{P}^n$ be a linear subspace of codimension 2 such that $L \cap X$ is smooth and L is not contained in X. Consider the *Lefschetz pencil* $P \simeq \mathbb{P}^1$ formed by all the hyperplanes $L \subset H \subset \mathbb{P}^n$. Then for almost all $H \in P$, the intersection $X \cap H$ is smooth.

Exercises in Algebraic Geometry 08.02.2022

Recall that $\mathrm{Pic}(C)$ is the quotient group of divisors on a curve C modulo the subgroup of principal divisors, and $\mathrm{Pic}^d(C) \subset \mathrm{Pic}(C)$ is the subset of degree d divisors.

1. Let $E \subset \mathbb{P}^2$ be a smooth *elliptic* curve cut out by the equation $y^2z - x^3 - axz^2 - bz^3 = 0$ (in particular, the polynomial $t^3 + at + b$ does not have multiple roots). We denote $e := (0,1,0) \in E$. We consider a map $\varphi\colon E \to \mathrm{Pic}^0(E),\ u \to [u - e]$. Prove that

a) φ is injective.

b) Given $u, v \in E$, there exists $w \in E$ such that $[u + v] = [w + e] \in \mathrm{Pic}^2(E)$.

c) For any effective divisor D on E of degree d, there is $w \in E$ such that

$$[D] = [w + (d-1)e] \in \mathrm{Pic}^d(E).$$

d) φ is a bijection of sets $E \xrightarrow{\sim} \mathrm{Pic}^0(E)$. It is used to equip $\mathrm{Pic}^0(E)$ with a structure of algebraic curve, and to equip E with a structure of algebraic group.

2. a) Prove that $\dim\Gamma(E,\mathcal{O}_E(ke))=k$ for any $k>0$.
b) Write down all the elements of $\Gamma(E,\mathcal{O}_E(ke))$ explicitly in the form $P(x)+yQ(x)$ for some $P,Q\in\mathsf{k}(x)$.

3. a) If $u+v=w$ in the sense of the group law of Problem 1 on E, write down the coordinates of w in terms of coordinates of u,v.
b) Prove that $u+v+w=0$ iff u,v,w lie on a line $\mathbb{P}^1\subset\mathbb{P}^2$.

4. Prove that if $u+v=0$, and u (resp. v) has coordinates $(x_1,y_1,1)$ (resp. $(x_2,y_2,1)$), then $x_1=x_2,\ y_1=-y_2$.

5. A point $u\in E$ is called an *inflection point* if the tangent line ℓ_u at u has the intersection order 3 (as opposed to the usual order 2) with E at u (in particular, $\ell_u\cap E=\{u\}$). Prove that
a) u is an inflection point on E iff u has order 3 in the group law of Problem 1 on E.
b) If a line $\ell\subset\mathbb{P}^2$ passes through two inflection points $u_1,u_2\in E$, then the third point of $\ell\cap E$ is an inflection point as well.

Exercises in Algebraic Geometry 15.02.2022

The sole purpose of this home assignment is to demonstrate the artificial difficulties caused by the absence of cohomology theory.

1. For a divisor D on a smooth projective curve C we define its *defect* $\operatorname{def}(D):=\deg(D)+1-\dim\Gamma(C,\mathcal{O}_C(D))$. By definition $\operatorname{def}(D)$ depends only on the class of D in $\operatorname{Pic}(C)$ (that is, modulo principal divisors). Prove that
a) If $D'-D$ is effective, then $\operatorname{def}(D')\geq\operatorname{def}(D)$.
b) There are divisors $D_k,\ k\in\mathbb{N}$, such that $\deg(D_k)\to\infty$, but $\operatorname{def}(D_k)\leq G$ for any k and some constant G.
c) Given a divisor D, for $k\gg 0$ we have $\dim\Gamma(C,\mathcal{O}_C(D_k-D))>0$.
d) For any divisor D, we have $\operatorname{def}(D)\leq G$.

The minimal constant G in d) is caled the *genus* $g=g(C)$. For a divisor D we set $h(D):=g-\operatorname{def}(D)=\dim\Gamma(C,\mathcal{O}_C(D))+g-1-\deg(D)$. By definition, $h(D)\geq 0$, and there is a divisor $D_{\min}$ such that $h(D_{\min})=0$.

2. Prove that a) for a point $c\in C$ and a divisor D we have $h(D)\geq h(D+c)\geq h(D)-1$.
b) There is a divisor D_0 of degree $g-1$ such that $h(D_0)=0$.
c) Any divisor of degree $d\geq g$ is equivalent (modulo principal divisors) to an effective divisor.
d) For any divisor D of degree $\deg(D)>2g-2$, we have $h(D)=0$.

3. Fix a point $c\in C$. We say that $k=1,2,3,\ldots$ is a *gap* for c if $\dim\Gamma(C,\mathcal{O}_C((k-1)c))=\dim\Gamma(C,\mathcal{O}_C(kc))$.
a) Prove that the number of gaps for c is finite. Compute this number.
b) Prove that $C\setminus\{c\}$ is an affine curve.

4. Let $\mathcal{L}$ be an invertible sheaf on C, and let D be an effective divisor with all multiplicities equal to 1. Then $\mathcal{L}(D)/\mathcal{L} = \bigoplus_{c\in D}(\mathcal{O}_C(c)\otimes\mathcal{L})|_c$ is a torsion sheaf with global sections $\Gamma(\mathcal{L}, D) = \bigoplus_{c\in D}(\mathcal{O}_C(c)\otimes\mathcal{L})|_c$. The exact sequence of sheaves

$$0 \to \mathcal{L} \to \mathcal{L}(D) \to \mathcal{L}(D)/\mathcal{L} \to 0$$

gives rise to $0 \to \Gamma(C,\mathcal{L}) \to \Gamma(C,\mathcal{L}(D)) \to \Gamma(\mathcal{L}, D)$. The cokernel of the rightmost morphism is denoted $H_D(\mathcal{L})$. Prove that $\dim H_D(\mathcal{L}) = h(\mathcal{L}) - h(\mathcal{L}(D))$ (it is understood that the function h is defined on $\mathrm{Pic}(C)$). In particular, for $\deg(D) \gg 0$ we have $h(\mathcal{L}(D)) = 0$, and $\dim H_D(\mathcal{L}) = h(\mathcal{L})$.

5. If $D' = D + D''$ are all effective multiplicity-free divisors, then we have a canonical embedding $\Gamma_D(\mathcal{L}) \hookrightarrow \Gamma_{D'}(\mathcal{L})$. Prove that

a) This embedding gives rise to a morphism $H_D(\mathcal{L}) \to H_{D'}(\mathcal{L})$.

b) The morphism $H_D(\mathcal{L}) \to H_{D'}(\mathcal{L})$ is always injective.

Now we define $H(\mathcal{L}) := \lim_D H_D(\mathcal{L})$. So for every (effective multiplicity free) D we have an embedding $H_D(\mathcal{L}) \hookrightarrow H(\mathcal{L})$, and this embedding is an isomorphism for $\deg(D) \gg 0$.

Exercises in Algebraic Geometry 22.02.2022

1. Let C be a smooth *hyperelliptic* projective curve with the field of rational functions $\mathsf{k}(C) = \mathsf{k}(x)[y]$, where $y^2 = (x-\lambda_1)\cdots(x-\lambda_{2g+2})$, and $\lambda_1,\ldots,\lambda_{2g+2} \in \mathsf{k}$ are all distinct.

a) Find the genus of C.

b) Write down an explicit basis in the space $\Gamma(C,\Omega_C)$ of regular differential forms on C (in coordinates x, y).

2. Let $\mathbb{P}^{g-1}$ be the projectivization of the dual space $\Gamma(C,\Omega_C)^\vee$. We have the canonical morphism $\phi\colon C \to \mathbb{P}^{g-1}$ (a point c goes to the hyperplane in the space of global sections formed by all the sections vanishing at c). Prove that ϕ decomposes into the 2-fold cover $\pi\colon C \to \mathbb{P}^1,\ (x,y) \mapsto x$, and the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$.

3. Prove that any curve of genus 1 or 2 is hyperelliptic (i.e. is a double cover of $\mathbb{P}^1$).

4. Prove that a) a smooth projective curve C is hyperelliptic iff there is an invertible sheaf $\mathcal{L}$ on C such that $\deg\mathcal{L} = 2$, and $\dim\Gamma(C,\mathcal{L}) \geq 2$.

b) If a smooth projective curve X is *not* hyperelliptic, then the canonical morphism $X \to \mathbb{P}^{g-1} = \mathbb{P}(\Gamma(X,\Omega_X)^\vee)$ is an embedding.

5. Let $X \subset \mathbb{P}^2$ be a smooth quartic (is cut out by a homogeneous polynomial of degree 4). Prove that X is *not* hyperelliptic, and the genus of X is 3.

Exercises in Algebraic Geometry 01.03.2022

1. We assume that the characteristic of k is 0. Let $S = \mathrm{Spec}(A)$ be a smooth affine curve. We have a natural permutation action of the symmetric group $S_n \curvearrowright A^{\otimes n}$. We set $\mathrm{Sym}^n(A) := (A^{\otimes n})^{S_n}$ (invariants of all permutations), and $\mathrm{Sym}^n(S) := \mathrm{Spec}(\mathrm{Sym}^n(A))$.

a) Prove that the points of $\mathrm{Sym}^n(S)$ are in natural bijection with the set of effective divisors of degree n on S.
b) Prove that $\mathrm{Sym}^n(S)$ is smooth.
c) Construct an algebraic variety $\mathrm{Sym}^n(C)$ with similar properties for a smooth *projective* curve C.

2. For a smooth projective curve C, we have a natural map

$$\pi_n\colon \mathrm{Sym}^n(C) \to \mathrm{Pic}^n(C),\ (c_1,\dots,c_n) \mapsto \mathcal{O}_C(c_1+\ldots+c_n).$$

Prove that
a) π_n is surjective for $n \geq g$ (the genus of C).
b) All the fibers of π_n are isomorphic to projective spaces $\mathbb{P}^k$ (for varying k).
c) All the fibers of π_n are of the same dimension for $n > 2g-2$.
d) There is an open dense subset $U \subset \mathrm{Sym}^g(C)$ such that $\pi_g|_U$ is injective.

3. Choose a point $c \in C$ and consider

$$\pi\colon \mathrm{Sym}^g(C) \to \mathrm{Pic}^0(C),\ (c_1,\dots,c_g) \mapsto \mathcal{O}_C(c_1+\ldots+c_g-gc).$$

According to the previous problem, $\pi|_U$ is injective. Define a structure of connected algebraic group on $\mathrm{Pic}^0(C)$ using the covering by $\pi(U)$ and its translates.

4. Prove that a) the group $\mathrm{Pic}^0(C)$ is generated by $\pi(U)$.
b) The group $\mathrm{Pic}^0(C)$ is the quotient of the free abelian group $\mathbb{Z}[\pi(U)]$ modulo the (obvious) relations $[u]-[u']-[u'']$ whenever $u = u'+u''$ for $u,u',u'' \in \pi(U)$.

5. Since $\mathrm{Pic}^n(C)$ is noncanonically isomorphic to $\mathrm{Pic}^0(C)$ (a choice of a point $\mathcal{L} \in \mathrm{Pic}^n(C)$ gives rise to such an isomorphism since $\mathrm{Pic}^n(C)$ is a principal homogeneous space over the group $\mathrm{Pic}^0(C)$), we obtain a structure of algebraic variety on $\mathrm{Pic}^n(C)$. Prove that the codifferential $d^*_{\pi_n}\colon T^*_{\mathcal{L}}\mathrm{Pic}^n(C) \to T^*_D\mathrm{Sym}^n(C)$ is injective for any $\mathcal{L} = \mathcal{O}_C(D) \in \mathrm{Pic}^n(C)$ for $n > 2g-2$.

Exercises in Algebraic Geometry 08.03.2022

The goal of this and next home assignments is to present an alternative construction of residues of differentials on curves, due to John Tate.

1. Let V be an (infinite dimensional, possibly) k-vector space (you should keep in mind a basic example $V = \mathsf{k}(\!(t)\!)$ of Laurent series field). Let $\mathrm{End}_f(V)$ denote the set of *finipotent* endomorphisms θ of V: such that for $n \gg 0$, the image of θ^n is finite dimensional. Prove that there is a unique map $\mathrm{Tr}_V\colon \mathrm{End}_f(V) \to \mathsf{k}$ such that
a) If $\dim V < \infty$, then Tr_V is the usual trace.
b) For a vector subspace $W \subset V$ and $\theta \in \mathrm{End}_f(V)$ such that $\theta W \subset W$, we have $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta) + \mathrm{Tr}_{V/W}(\theta)$.
c) If θ is nilpotent, $\mathrm{Tr}_V(\theta) = 0$.

d) For a finipotent linear subspace $F \subset \mathrm{End}_f(V)$ (i.e. for $n \gg 0$, and any $\theta_1, \ldots, \theta_n \in F$, $\dim \theta_1 \cdots \theta_n V < \infty$), the trace $\mathrm{Tr}_V \colon F \to \mathsf{k}$ is k-linear.

e) If $\varphi \colon V' \to V$ and $\psi \colon V \to V'$ are k-linear operators, and $\varphi\psi$ is finipotent, then $\psi\varphi$ is finipotent, and $\mathrm{Tr}_V(\varphi\psi) = \mathrm{Tr}_{V'}\psi\varphi$.

2. We say that a subspace $A \subset V$ is *not much bigger* than a subspace $B \subset V$ if $\dim(A+B)/B < \infty$; then we write $A \precsim B$. We say that A, B are *about the same size* if $A \precsim B$ and $B \precsim A$; then we write $A \sim B$. You should keep in mind a basic example $A = t^m \mathsf{k}[\![t]\!]$, $B = t^n \mathsf{k}[\![t]\!]$.

Given $A \subset V$ we define the following subspaces $E, E_0, E_1, E_2 \subset \mathrm{End}(V)$ as follows: $\theta \in E$ iff $\theta A \precsim A$, and $\theta \in E_1$ iff $\theta V \precsim A$, and $\theta \in E_2$ iff $\theta A \precsim 0$; finally, $E_0 = E_1 \cap E_2$. Prove that

a) E is a subalgebra of $\mathrm{End}(V)$, and E_i are two-sided ideals in E; they all depend only on the $\sim$-equivalence class of A.

b) $E_1 \cap E_2 = E_0$, $E_1 + E_2 = E$, and E_0 is finipotent.

c) Let $\varphi \in E_0$, $\psi \in E$. Then $[\varphi, \psi] = \varphi\psi - \psi\varphi \in E_0$, and $\mathrm{Tr}_V[\varphi, \psi] = 0$.

d) Let $\varphi \in E_1$, $\psi \in E_2$. Then $[\varphi, \psi] = \varphi\psi - \psi\varphi \in E_0$, and $\mathrm{Tr}_V[\varphi, \psi] = 0$.

3. Let K be a commutative k-algebra (you should keep in mind a basic example $K = \mathsf{k}(\!(t)\!)$ of Laurent series field), V a K-module, and A a k-subspace of V such that $fA \precsim A$ for any $f \in K$. Hence we have a morphism $K \to E \subset \mathrm{End}(V)$. Recall that we have a surjective map $c \colon K \otimes_{\mathsf{k}} K \twoheadrightarrow \Omega_{K/\mathsf{k}}$, $c(f \otimes g) = f dg$, and $\mathrm{Ker}(c)$ is generated over k by elements of the form $f \otimes gh - fg \otimes h - fh \otimes g$.

Prove that a) there is a unique k-linear *residue* map $\mathrm{Res}^V_A \colon \Omega_{K/\mathsf{k}} \to \mathsf{k}$ such that for any $f, g \in K$ we have $\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_V[f', g']$, where $f', g' \in E$ are endomorphisms such that $f \equiv f' \pmod{E_2}$, $g \equiv g' \pmod{E_2}$, and either $f' \in E_1$ or $g' \in E_1$.

b) If we set $B = A + gA$, $C = B \cap f^{-1}(A) \cap (fg)^{-1}(A)$, and π is a k-linear projection of $A + fA + fgA$ onto A, then $\dim(B/C) < \infty$, and $\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_{B/C}[\pi f, g]$.

4. Prove that a) if $V \supset V' \supset A$ and $KV' = V'$, then $\mathrm{Res}^V_A = \mathrm{Res}^{V'}_A$. For this reason we will often write Res_A for Res^V_A.

b) If $A \sim A'$, then $\mathrm{Res}^V_A = \mathrm{Res}^V_{A'}$.

c) If $fA + fgA + fg^2A \subset A$, then $\mathrm{Res}_A(f dg) = 0$. In particular, $\mathrm{Res}_A \equiv 0$ if A is a K-submodule of V.

d) For $g \in K$ and $n \in \mathbb{N}$ we have $\mathrm{Res}_A(g^n dg) = 0$.

e) If $g \in K$ is invertible, then $\mathrm{Res}_A(g^n dg) = 0$ for any $n \leq -2$.

5. Prove that a) if $g \in K$ is invertible, and for $h \in K$ we have $hA \subset A$, then $\mathrm{Res}_A(hg^{-1} dg) = \mathrm{Tr}_{A/(A \cap gA)}(h) - \mathrm{Tr}_{gA/(A \cap gA)}(h)$. In particular, if g is invertible, and $gA \subset A$, then $\mathrm{Res}_A(g^{-1} dg) = \dim_{\mathsf{k}}(A/gA)$.

b) For another subspace $B \subset V$ such that $fB \precsim B$ for any $f \in K$, we have $f(A + B) \precsim A + B$, $f(A \cap B) \precsim A \cap B$ for any $f \in K$, and $\mathrm{Res}_A + \mathrm{Res}_B = \mathrm{Res}_{A+B} + \mathrm{Res}_{A \cap B}$.

c) For a commutative K-algebra K' free of finite rank over K, setting $V' = K' \otimes_K V$ and $A' = \sum_i x_i \otimes A$ (for some basis $\{x_i\}$ of K' over K), the $\sim$-equivalence class of A' depends only on A, and $f'A' \precsim A'$ for any $f' \in K'$. Furthermore, $\mathrm{Res}_{A'}(f'dg) = \mathrm{Res}_A((\mathrm{Tr}_{K'/K} f)dg)$ for any $f' \in K'$ and $g \in K$.

Exercises in Algebraic Geometry 15.03.2022

Let X be a smooth projective curve, and $K = \mathsf{k}(X)$. For a point $p \in X$ we denote by $\mathcal{O}_x \subset K$ its local ring, and by $A_p = \widehat{\mathcal{O}}_x$ its completion. Finally, $K_p = \mathrm{Frac}\, A_p$ is the completion of K at p. We define the residue $\mathrm{Res}_p \colon \Omega_{K/\mathsf{k}} \to \mathsf{k}$ by $\mathrm{Res}_p f dg := \mathrm{Res}^{K_p}_{A_p}(fdg)$ (notation of the previous home assignment).

1. Let us choose a uniformizer $t \in A_p$, so that $A_p \simeq \mathsf{k}[\![t]\!]$, and $K_p \simeq \mathsf{k}(\!(t)\!)$. Let $f = \sum_{n \geq -N} a_n t^n$, $g = \sum_{m \geq -M} b_m t^m \in K_p$. Prove that $\mathrm{Res}_p f dg =$ the coefficient of t^{-1} in $f(t)\frac{d}{dt}g(t)$, that is $\sum_{n+m=0} m a_n b_m$.

2. Let S be any subset of X. We set $\mathcal{O}(S) := \bigcap_{p \in S} \mathcal{O}_p \subset K$, $A_S := \prod_{p \in S} A_p$, $V_S := \prod'_{p \in S} K_p$ (the set of collections $(f_p \in K_p)_{p \in S}$ such that $f_p \in A_p$ for almost all $p \in S$). In particular, we have a diagonal embedding $K \subset V_S$.

a) Prove that $\dim(V_S/(K + A_S)) < \infty$; in particular, $V_X/(K + A_X) = H^1(X, \mathcal{O}_X)$.

b) Prove that $\mathrm{Res}_{A_S} + \mathrm{Res}_K = \mathrm{Res}_{\mathcal{O}(S)} + \mathrm{Res}_{K+A_S}$.

c) Let $f, g \in K$, and let $S' \subset S$ be a *finite* subset. Set $T := S \setminus S'$. Prove that $\mathrm{Res}_{A_S} f dg = Res_{A_T} f dg + \sum_{p \in S'} \mathrm{Res}_p f dg$.

d) Prove that for $\omega \in \Omega_{K/\mathsf{k}}$, we have $\sum_{p \in S} \mathrm{Res}_p \omega = \mathrm{Res}^K_{\mathcal{O}(S)} \omega$.

e) Prove that for $\omega \in \Omega_{K/\mathsf{k}}$, we have $\sum_{p \in X} \mathrm{Res}_p \omega = 0$.

3. Let $S \subset X$ be a finite subset. Let $\mathcal{F}$ be a locally free coherent sheaf on X.

a) Prove that a collection of principal parts $(\omega_p)_{p \in S}$ comes from a rational form $\omega \in \Omega_{K/\mathsf{k}}$ regular on $X \setminus S$ iff $\sum_{p \in S} \mathrm{Res}_p \omega_p = 0$.

b) Prove that a collection of principal parts $(\phi_p)_{p \in S}$ comes from a rational section $\phi \in \mathrm{Rat}(\mathcal{F})$ regular on $X \setminus S$ iff $\sum_{p \in S} \mathrm{Res}_p \langle \phi_p, \varpi \rangle = 0$ for all $\varpi \in \Gamma(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \Omega_X)$.

4. Let $\pi \colon Y \to X$ be a dominant morphism of smooth projective curves corresponding to the inclusion of function fields $K \subset L$. Prove that

a) For $f \in L$, $g \in K$, $p \in X$, we have $\sum_{\pi(y)=p} \mathrm{Res}_y f dg = \mathrm{Res}_p((\mathrm{Tr}_{L/K} f)dg)$.

b) For $\pi(y) = p$, $g \in K_p$, $f \in L_y$, we have $\mathrm{Res}_y f dg = \mathrm{Res}_p((\mathrm{Tr}_{L_y/K_p} f)dg)$.

5. Let X, Y be smooth projective curves, and the genus $g(X) > 1$. Prove that

a) There are finitely many dominant morphisms from Y to X.

b) The order of the finite group $\mathrm{Aut}(X)$ is at most $84(g(X) - 1)$.

Exercises in Algebraic Geometry 22.03.2022

1. Let k have characteristic 3. Let $X \subset \mathbb{P}^2$ be the curve cut out by equation $x^3y + y^3z + z^3x = 0$. Prove that X is smooth, but any point $x \in X$ is an inflection point.

2. Prove that the dual curve $X^\vee$ in the dual projective space is isomorphic to X, and the Gauß map $g\colon X \to X^\vee$ (a point $x \in X$ goes to the tangent line to X at x) is purely inseparable.

3. Prove that the group of automorphisms of X is isomorphic to $\mathrm{PSL}(2, \mathbb{F}_7)$.

4. For a smooth curve $C \subset \mathbb{P}^2$ of degree $d > 1$ such that the Gauß map $g\colon C \to C^\vee$ is birational, find the degree of $C^\vee$.

5. Let $\pi\colon E_1 \to E_2$ be a homomorphism from an elliptic curve to another elliptic curve (dominant morphism of degree d). The pullback of invertible sheaves gives rise to a homomorphism $\pi^*\colon E_2 = \mathrm{Pic}^0(E_2) \to \mathrm{Pic}^0(E_1) = E_1$. Prove that
a) $\pi \circ \pi^* = d$ (multiplication by d in the group E_2).
b) If $E_1 = E_2$, and $\pi = c$ (multiplication by c in the group E_1), then $\pi^* = c$ as well.
c) $\pi_1^* \circ \pi_2^* = (\pi_2 \circ \pi_1)^*$.
d) $(\pi^*)^* = \pi$.
e) $(\pi + \varpi)^* = \pi^* + \varpi^*$, if $\pi + \varpi \neq 0$.

Exercises in Algebraic Geometry 29.03.2022

1. Let $Q \subset \mathbb{P}^3$ be a smooth quadric, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Segre embedding). Prove that $\mathrm{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}$, that is any line bundle on Q is isomorphic to $\mathcal{O}_Q(a, b) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(b)$, and any divisor is of type (a, b).

2. Prove that a) if $|a - b| \leq 1$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$.
b) If $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$.
c) If $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$.
d) Any effective divisor of type (a, b) for $a, b > 0$ is connected.

3. Prove that for any $a, b > 0$ there is a smooth connected curve $X \subset Q$ of type (a, b).

4. Prove that $g(X) = ab - a - b + 1$. In particular, if $(a, b) = (g + 1, 2)$, we get a smooth connected curve X of genus g.

5. Prove that the above curve $X \subset Q$ is *projectively normal* (i.e. the restriction map from the homogeneous coordinate ring of $\mathbb{P}^3$ to the homogeneous coordinate ring of $X \subset \mathbb{P}^3$ is surjective) iff $|a - b| \leq 1$. In particular a smooth connected curve X of type $(1, 3)$ (rational curve of degree 4 in $\mathbb{P}^3$) is *not* projectively normal.

Exercises in Algebraic Geometry 05.04.2022

1. Let $\overline{X}$ be a smooth projective curve of genus g over $\mathsf{k} = \overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$. Let N_r be the number of points of $\overline{X}$, rational over $\mathbb{F}_{q^r}$. Let $|X|$ be the set of Fr-orbits in $\overline{X}$, and for $x \in |X|$ let $\deg(x)$ be the cardinality of the orbit. We define the zeta-function $\zeta_X(s) = Z(X; t = q^{-s})$ as $\prod_{x \in |X|}(1 - t^{\deg(x)})^{-1}$. Prove that
a) $Z(X; t) = \exp(\sum_{r=1}^{\infty} N_r t^r / r)$.

b) $Z(X;t) = \sum_D t^{\deg D}$, where the sum runs over all the effective divisors on $\overline{X}$ defined over $\mathbb{F}_q$, that is over the positive linear combinations of Fr-orbit sums in $|X|$.

2. Prove that a) for a given effective divisor D defined over $\mathbb{F}_q$, the number of all the effective divisors defined over $\mathbb{F}_q$ and linearly equivalent to D equals $(q^{h^0(D)} - 1)/(q-1)$, where $h^0(D) := \dim H^0(\overline{X}, \mathcal{O}_{\overline{X}}(D))$.

b) The number of linear equivalence classes of degree 0 divisors on $\overline{X}$ defined over $\mathbb{F}_q$ is finite. It is denoted h.

c) It is known that there is a divisor of degree 1 defined over $\mathbb{F}_q$ (you can try to prove this fact, but it is not trivial). Deduce that the number of linear equivalence classes of degree n divisors on $\overline{X}$ defined over $\mathbb{F}_q$ is equal to h for any degree n.

3. Grouping the summands in Problem 1b) by the linear equivalence classes (and recalling that for $\deg D > 2g-2$ we have $h^0(D) = \deg D + 1 - g$), prove that $Z(X;t) = P_1(t)P_0(t)^{-1}P_2(t)^{-1}$, where $P_0(t) = 1-t,\ P_2(t) = 1-qt,\ P_1(t) = \prod_{i=1}^{2g}(1-\alpha_i t)$.

4. Deduce from the Serre duality that $Z(X;(qt)^{-1}) = q^{1-g}t^{2-2g}Z(X;t)$.

5. a) In notation of Problem 1, apply the Castelnuovo-Severi inequality to the graphs of the identity morphism $\mathrm{Id}\colon \overline{X} \to \overline{X}$ and of the Frobenius morphism $\mathrm{Fr}^r\colon \overline{X} \to \overline{X}$ and deduce $N_r = 1 - a_r + q^r$ for $|a_r| \leq 2g\sqrt{q^r}$.

b) In notation of Problem 3, prove $a_r = \sum_{i=1}^{2g} \alpha_i^r$.

c) Prove that the condition $a_r \leq 2g\sqrt{q^r}\ \forall r$ is equivalent to the condition $|\alpha_i| \leq \sqrt{q}\ \forall i$.

d) Prove that $|\alpha_i| = \sqrt{q}\ \forall i$ (Riemann hypothesis for the curve $\overline{X}$).

Exercises in Algebraic Geometry 12.04.2022

1. Let X be a smooth projective irreducible surface. Recall that for $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(X)$ we set

$$[\mathcal{L}, \mathcal{M}] := \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1}).$$

We consider the quadratic form $Q(\mathcal{M}) := [\mathcal{M}, \mathcal{M}]$ on $\mathrm{Pic}(X)$. Prove that

a) If $X = C \times S$, then $N(\mathcal{M}) := \chi_C(\mathcal{M})\chi_S(\mathcal{M}) - \chi_X(\mathcal{M}) = -\frac{1}{2}Q(\mathcal{M}) + \deg_C \mathcal{M} \cdot \deg_S \mathcal{M}$.

b) $Q(\mathcal{M}) = 2 \lim\limits_{n\to\infty} (\chi(\mathcal{M}^{\otimes n})/n^2)$.

2. Fix a very ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. For $\mathcal{M} \in \mathrm{Pic}(X)$ we set $d(\mathcal{M}) := [\mathcal{L}, \mathcal{M}]$. Prove that if $d(\mathcal{M}) < 0$, then $\Gamma(X, \mathcal{M}) = 0$.

3. Choose a section $\sigma \in \Gamma(X, \mathcal{L})$ such that its zero divisor D is a smooth curve (perhaps disconnected). Prove that

a) For any $\mathcal{M} \in \mathrm{Pic}(X)$, we have $\dim \Gamma(X, \mathcal{M}) \leq \dim \Gamma(X, \mathcal{M}(-D)) + \max(d(\mathcal{M}), 0) + c$ for a constant c.

b) There is a function $\Phi(n)$ on $\mathbb{Z}$ such that for any $\mathcal{M} \in \mathrm{Pic}(X)$ we have $\dim \Gamma(X, \mathcal{M}) \leq \Phi(d(\mathcal{M}))$.

4. Deduce from the Serre duality on X (perfect pairing between $H^{2-i}(X, \omega_X \otimes \mathcal{M}^{-1})$ and $H^i(X, \mathcal{M})$), that there is a function $\Psi(n)$ on $\mathbb{Z}$ such that for any $\mathcal{M} \in \operatorname{Pic}(X)$ we have $\dim H^2(X, \mathcal{M}) \leq \Psi(d(\mathcal{M}))$.

5. Take $\mathcal{M} \in \operatorname{Pic}(X)$ with $d(\mathcal{M}) = 0$. Prove that
a) $\chi(\mathcal{M}^{\otimes n})$ is bounded from above.
b) $\chi(\mathcal{M}^{\otimes n}) = \frac{1}{2}n^2 Q(\mathcal{M})$ plus a linear function in n.
c) $Q(\mathcal{L}) > 0$.
d) If $d(\mathcal{M}) = 0$, then $Q(\mathcal{M}) \leq 0$ (*Hodge index theorem*).

Exercises in Algebraic Geometry 19.04.2022

1. Let $m \in \mathbb{Z}$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for any $i > 0$.
a) Find all m such that $\mathcal{O}_{\mathbb{P}^n}(r)$ is m-regular.
b) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of coherent sheaves on $\mathbb{P}^n$. Prove that if $\mathcal{F}', \mathcal{F}''$ are both m-regular, then $\mathcal{F}$ is m-regular as well.
c) Prove that if $\mathcal{F}'$ is $(m+1)$-regular, and $\mathcal{F}$ is m-regular, then $\mathcal{F}''$ is m-regular.
d) Prove that if $\mathcal{F}''$ is $(m-1)$-regular, and $\mathcal{F}$ is m-regular, then $\mathcal{F}'$ is m-regular.
e) Let $S^0_{\mathcal{F}} \subset \mathbb{P}^n$ be the union of supports of all the sky-scraper subsheaves of $\mathcal{F}$ (so that $S^0_{\mathcal{F}}$ is a finite subset of $\mathbb{P}^n$). Let $S^1_{\mathcal{F}}$ be the (finite) set of curves in $\mathbb{P}^n$ equal to the 1-dimensional supports of subsheaves of $\mathcal{F}$. We define $S^2_{\mathcal{F}}, \ldots, S^{n-1}_{\mathcal{F}}$ similarly, and $S_{\mathcal{F}} := S^0_{\mathcal{F}} \cup \ldots \cup S^{n-1}_{\mathcal{F}}$ (it is not a subvariety of $\mathbb{P}^n$, but a collection of closed subvarieties). Assume a hyperlane $H \subset \mathbb{P}^n$ does not contain any closed subvariety in $S_{\mathcal{F}}$. Prove that if $\mathcal{F}$ is m-regular, then the restriction $\mathcal{F}|_H$ is also m-regular.

2. Assume a hyperplane $H \subset \mathbb{P}^n$ does not contain any closed subset in $S_{\mathcal{F}}$. Prove $\operatorname{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{F}) = 0$.

3. Let $\mathcal{F}$ be an m-regular coherent sheaf on $\mathbb{P}^n$. Prove that $\mathcal{F}$ is k-regular for any $k \geq m$.

4. Prove that the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \Gamma(\mathbb{P}^n, \mathcal{F}(r)) \to \Gamma(\mathbb{P}^n, \mathcal{F}(r+1))$ is surjective for any $r \geq m$.

5. a) Assume that the restriction $\mathcal{F}|_H$ to a hyperplane $H \subset \mathbb{P}^n$ is r-regular, and the restriction morphism $\Gamma(\mathbb{P}^n, \mathcal{F}(r)) \to \Gamma(H, \mathcal{F}(r)|_H)$ is surjective. Prove that the restriction morphism $\Gamma(\mathbb{P}^n, \mathcal{F}(r+1)) \to \Gamma(H, \mathcal{F}(r+1)|_H)$ is surjective as well.
b) Prove that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(r)$ is generated by its global sections, and $H^{>0}(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for any $r \geq m$.

Exercises in Algebraic Geometry 26.04.2022

1. Let $S = \mathsf{k}[x_1, \ldots, x_n]$, and let $\Lambda^\bullet = \Lambda^\bullet\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ be the exterior algebra on n generators. We consider the *Koszul complex* $K^\bullet = S \otimes \Lambda^{\bullet+n}$ living in cohomological degrees

$-n,\ldots,0$. It has many equivalent definitions; choose your favorite one. For instance, it is the n-th tensor power of the complex $\mathsf{k}[x] \xrightarrow{x} \mathsf{k}[x]$ living in cohomological degrees $-1,0$. Prove that its cohomology is k in degree 0, i.e. it gives a length n free resolution of the augmentation S-module k.

2. We say that a complex $M^\bullet$ of graded S-modules is *minimal* if all the differentials in $M^\bullet \otimes_S \mathsf{k}$ are zero. For instance, the Koszul complex is minimal. Prove that a free resolution $\ldots \xrightarrow{d_{-2}} M^{-1} \xrightarrow{d_{-1}} M^0$ of a graded S-module $H^0 := \mathrm{Coker}(d_{-1})$ is minimal iff for any $i<0$, a basis of M^{i+1} projects to some minimal collection of generators of $\mathrm{Coker}(d_i)$.

3. We define the *projective dimension* $\mathrm{pdim}(H)$ of a finitely generated graded S-module H as the minimal length of a projective resolution of H. Prove that $\mathrm{pdim}(H)$ equals the length of any minimal free resolution of H.

4. Prove that $\mathrm{pdim}(H)$ is the minimal $\ell \in \mathbb{N}$ such that $\mathrm{Tor}^S_{\ell+1}(\mathsf{k}, H) = 0$. In particular, any finitely generated graded S-module H admits a free resolution of length at most n (since you may compute Tor via the Koszul resolution of k).

5. Prove that a) any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ admits a resolution

$$0 \to \mathcal{M}^{-n-1} \to \ldots \to \mathcal{M}^0 \to \mathcal{F} \to 0$$

such that any $\mathcal{M}^i$ is a direct sum of invertible sheaves.

b) Any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ admits a resolution

$$0 \to \mathcal{M}^{-n} \to \ldots \to \mathcal{M}^0 \to \mathcal{F} \to 0$$

such that any $\mathcal{M}^i$ is a locally free sheaf.

c) The Euler characteristic $\chi(\mathcal{F}(r))$ is a polynomial in r (the *Hilbert polynomial*).

d) This Hilbert polynomial coincides with P_M of Problem 3 of December 02, 2021, where M is a graded $\mathsf{k}[x_0,\ldots,x_n]$-module corresponding to $\mathcal{F}$.

Exercises in Algebraic Geometry 03.05.2022

1. Consider the rational morphism

$$\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^4,\ (x_0:x_1:x_2) \mapsto (x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

Prove that ϕ is birational isomorphism onto the image closure $Y \subset \mathbb{P}^4$, and the inverse rational morphism $\phi^{-1}\colon Y \dashrightarrow \mathbb{P}^2$ is actually regular, and is nothing but the blowup of $\mathbb{P}^2$ at the point $(1:0:0)$.

2. Let $\psi\colon Q \dashrightarrow H$ be the stereographic projection of a smooth quadric $Q \subset \mathbb{P}^3$ from a point $q \in Q$ to a hyperplane $H \simeq \mathbb{P}^2 \subset \mathbb{P}^3$. Decompose $\psi = \varphi \circ \phi^{-1}$ for regular morphisms

$$Q \xleftarrow{\phi} X \xrightarrow{\varphi} H,$$

where ϕ and φ are blowups (perhaps at several points).

3. Let $x, y \in L \subset \mathbb{P}^2$ be two distinct points of a projective line. Let Z be the blowup of $\mathbb{P}^2$ at x, y. Let $\widetilde{L} \subset Z$ be the strict transform of L (the irreducible curve equal to the closure of the preimage of $L \setminus \{x, y\}$).

a) Prove that the self-intersection index $I(E, E)$ for any exceptional divisor $E \simeq \mathbb{P}^1 \subset Z$ (defined e.g. as $[\mathcal{O}_Z(E), \mathcal{O}_Z(E)]$, notation of Problem 1 of April 12) is -1.

b) Prove that the self-intersection index $I(\widetilde{L}, \widetilde{L})$ is -1.

c) By Castelnuovo theorem, $\widetilde{L} \subset Z$ can be blown down. Describe explicitly the surface W obtained by this blow-down.

4. Let $\pi \colon \widetilde{X} \to X$ be the blowup of a smooth irreducible projective surface X at a point $x \in X$, and let $E = \pi^{-1}(x) \subset \widetilde{X}$ be the exceptional divisor. Recall that the intersection index $E \cdot E := [\mathcal{O}_{\widetilde{X}}(E), \mathcal{O}_{\widetilde{X}}(E)] = -1$. We have the homomorphisms $\pi^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}),\ a \mapsto \mathcal{O}_{\widetilde{X}}(aE)$. Prove that

a) These homomorphisms give rise to $\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}$.

b) $[\pi^*\mathcal{L}, \pi^*\mathcal{M}] = [\mathcal{L}, \mathcal{M}]$ for any $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(X)$.

c) $[\pi^*\mathcal{L}, \mathcal{O}_{\widetilde{X}}(E)] = 0$ for any $\mathcal{L} \in \mathrm{Pic}(X)$.

d) $[\pi^*\mathcal{L}, \mathcal{N}] = [\mathcal{L}, \pi_*\mathcal{N}]$ for any $\mathcal{L} \in \mathrm{Pic}(X)$, $\mathcal{N} \in \mathrm{Pic}(\widetilde{X})$, where $\pi_*\mathcal{N}$ stands for the projection to the direct summand $\mathrm{Pic}(X)$ of $\mathrm{Pic}(\widetilde{X})$.

5. Prove that a) $\omega_{\widetilde{X}} \simeq \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(E)$.

b) $[\omega_{\widetilde{X}}, \omega_{\widetilde{X}}] = [\omega_X, \omega_X] - 1$.

Exercises in Algebraic Geometry 10.05.2022

1. For an effective divisor $D \subset X$, the *linear system* $|D|$ is the set of all effective divisors linearly equivalent to D (i.e. zero divisors of sections $\sigma \in \Gamma(X, \mathcal{O}_X(D))$). Given points $P_1, \ldots, P_r \in X$, the linear subsystem $|D - P_1 - \ldots - P_r|$ *with prescribed base points* is formed by the zero divisors of all the sections $\sigma \in \Gamma(X, \mathcal{O}_X(D))$ vanishing at $P_1, \ldots, P_r$. If all the divisors in $|D - P_1 - \ldots - P_r|$ contain a common point $P \in X$, then this P is called a *non-prescribed base point* of the linear system $|D - P_1 - \ldots - P_r|$.

Consider points $P_1, \ldots, P_r \in \mathbb{P}^2$ such that no 3 of them are collinear. Prove that the linear system of conics (i.e. zero divisors of sections of $\mathcal{O}_{\mathbb{P}^2}(2)$) with presribed base points $P_1, \ldots, P_r$ has no non-prescribed base points if $r \leq 4$.

2. Prove that a) the dimension of the linear system of conics with prescribed base points $P_1, \ldots, P_r$ equals $5 - r$ for any $r \leq 5$.

b) There exists a unique conic passing through $P_1, \ldots, P_5$ (recall that no 3 points are collinear), and this conic is necessarily irreducible.

3. Consider distinct points $P_1, \ldots, P_r \in \mathbb{P}^2$ such that no 4 of them are collinear and no 7 of them lie on a common conic. Prove that the linear system of cubics (i.e. zero divisors of

sections of $\mathcal{O}_{\mathbb{P}^2}(3)$) with presribed base points $P_1, \ldots, P_r$ has no non-prescribed base points if $r \leq 7$.

4. Prove that a) the dimension of the linear system of cubics with prescribed base points $P_1, \ldots, P_r$ equals $9 - r$ for any $r \leq 8$.

b) There is a 1-dimensional linear system of cubics passing through $P_1, \ldots, P_8$, and all but finitely many cubics in this linear system are irreducible.

5. Prove that for distinct points $P_1, \ldots, P_8 \in \mathbb{P}^2$ (such that no 4 of them are collinear and no 7 of them lie on a common conic) there is a unique point P_9 such that any cubic curve passing through $P_1, \ldots, P_8$ contains P_9.

Exercises in Algebraic Geometry 17.05.2022

1. Consider points $P_1, \ldots, P_r \in \mathbb{P}^2$ such that no 3 of them are collinear and no 6 of them lie on a common conic. The linear system of cubics with prescribed base points $P_1, \ldots, P_r$ is in a natural bijection with the linear system $|\pi^*\mathcal{O}_{\mathbb{P}^2}(3) - E_1 - \ldots - E_r|$ on the blowup $\widetilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$ at $P_1, \ldots, P_r$; namely, $\mathbb{P}^2 \supset D \mapsto \pi^{-1}(D) - E_1 - \ldots - E_r$, where $E_1, \ldots, E_r$ are the exceptional divisors. Prove that the latter linear system is very ample if $r \leq 6$.

To this end, you will have to apply the criterion of very ampleness. The infinitesimal condition in this criterion has to be reformulated as follows: for any point $P \in E_i$ (*infinitesimally close* to P_i), the latter linear system with prescribed base point P has no non-presribed base points.

2. Prove that a) $\widetilde{\mathbb{P}}^2$ embeds into $\mathbb{P}^{9-r}$ as a surface of degree $d = 9 - r$ (*del Pezzo surface*).

b) The dualizing sheaf $\omega_{\widetilde{\mathbb{P}}^2} \simeq \mathcal{O}_{\widetilde{\mathbb{P}}^2}(-1)$. In particular, for $r = 6$, $\widetilde{\mathbb{P}}^2$ is a smooth cubic hypersurface in $\mathbb{P}^3$. Counting the parameters and comparing with Problem 4c) of December 16th, we conclude that a general cubic surface in $\mathbb{P}^3$ is obtained as the blowup of $\mathbb{P}^2$ at 6 points. In fact, this is true for *any* smooth cubic surface in $\mathbb{P}^3$.

3. We fix 6 distinct points $P_1, \ldots, P_6 \in \mathbb{P}^2$ not lying on a common conic, such that no 3 are collinear. We denote by $X \subset \mathbb{P}^3$ the blowup of $\mathbb{P}^2$ at these points, and we denote by $E_1, \ldots, E_6 \subset X$ the exceptional divisors. We fix a line $L \subset \mathbb{P}^2$ not passing through any of P_i, and keep the name $L \subset X$ for its preimage in X. Prove that

a) $\mathrm{Pic}(X) = \mathbb{Z}^7$ is generated by $\mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_6), \mathcal{O}_X(L)$.

b) $[\mathcal{O}_X(E_i), \mathcal{O}_X(E_i)] = -1$, $[\mathcal{O}_X(E_i), \mathcal{O}_X(E_j)] = 0 = [\mathcal{O}_X(E_i), \mathcal{O}_X(L)]$, $[\mathcal{O}_X(L), \mathcal{O}_X(L)] = 1$ for any $i \neq j$.

c) $\mathcal{O}_X(1) \simeq \mathcal{O}_X(3L - E_1 - \ldots - E_6)$.

d) $\omega_X \simeq \mathcal{O}_X(-1)$.

4. Let $D \subset X$ be an effective divisor such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(aL - \sum_{i=1}^6 b_i E_i)$. Prove that

a) The degree of D is $d = 3a - \sum_{i=1}^6 b_i$.

b) $[\mathcal{O}_X(D), \mathcal{O}_X(D)] = a^2 - \sum_{i=1}^6 b_i^2$.

c) The arithmetic genus $p_a(D) = \frac{1}{2}[\mathcal{O}_X(D), \mathcal{O}_X(D)] - \frac{1}{2}d + 1 = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum_{i=1}^{6} b_i(b_i - 1)$.

5. Prove that a) X contains exactly 27 lines $\mathbb{P}^1 \subset X \subset \mathbb{P}^3$. Namely,
i) The exceptional divisors $E_1, \ldots, E_6$;
ii) The strict transforms F_{ij} of the lines in $\mathbb{P}^2$ through P_i and P_j;
iii) The strict transforms G_i of conics $C_i \subset \mathbb{P}^2$ passing through a quintuple of our points.
b) For any line $\mathfrak{L}$ of those 27 lines, we have $[\mathcal{O}_X(\mathfrak{L}), \mathcal{O}_X(\mathfrak{L})] = -1$.
c) Any irreducible curve $D \subset X$ such that $[\mathcal{O}_X(D), \mathcal{O}_X(D)] < 0$ is one of those 27 lines.

Exercises in Algebraic Geometry 24.05.2022

1. a) Find the explicit equations of the lines $E_i, F_{ij}, G_i,\ 1 \leq i \neq j \leq 6$, on the Fermat cubic surface cut out by the equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.
b) Check that $E_i \cap E_j = \emptyset,\ G_i \cap G_j = \emptyset$ for $i \neq j$; furthermore, $E_i \cap F_{jk} = \emptyset,\ G_i \cap F_{jk} = \emptyset$ iff $j \neq i \neq k$; furthermore, $E_i \cap G_j = \emptyset$ iff $i = j$; finally, $F_{ij} \cap F_{kl} = \emptyset$ iff $\{i, j\} \cap \{k, l\} \neq \emptyset$.

2. Let C be an irreducible cubic curve in $\mathbb{P}^2$. Let a line L (resp. L') intersect C in the points P, Q, R (resp. P', Q', R'). Let P'' (resp. Q'', R'') be the third point of the intersection of the line through P, P' (resp. through Q, Q' and R, R') with C. Prove that P'', Q'', R'' lie on a common line.

3. Let P_0 be an inflection point of C. We define the following group operation on the set of smooth points of C: given smooth points $P, Q \in C$, we define $R \in C$ as the third point of the intersection of C with the line through P, Q, and then we define $T \in C$ as the third point of the intersection of C with the line through P_0, R. We set $P + Q := T$. Deduce from the previous problem that this operation is associative.

4. Prove the famous Pascal theorem: Let A, B, C, A', B', C' be a sextuple of points on a smooth conic in $\mathbb{P}^2$. Let P (resp. Q, R) be the intersection of the lines through A, B' and A', B (resp. through A, C' and A', C; through B, C' and B', C). Prove that P, Q, R lie on a common line.

5. Fix points $P_1, \ldots, P_{13}$ in $\mathbb{P}^2$ in general position.
a) Prove that there are P_{14}, P_{15}, P_{16} such that any quartic containing $P_1, \ldots, P_{13}$ necessarily contains P_{14}, P_{15}, P_{16} as well.
b) Formulate explicitly the "general position" condition (like no 4 points are collinear, etc.)

Exercises in Algebraic Geometry 31.05.2022

1. For a smooth irreducible algebraic variety X we set $\Omega_X^i := \Lambda^i \Omega_X$ (higher order differential forms). In particular, $\Omega_X^{\dim X} = \omega_X$ is the dualizing sheaf. Prove that $\dim \Gamma(X, \Omega_X^i)$ is a birational invariant of X, i.e. if projective varieties X and Y are birationally isomorphic,

then $\dim \Gamma(X, \Omega^i_X) = \dim \Gamma(Y, \Omega^i_Y)$ (you may use the Hartogs theorem and the theorem about extension through the generic points of divisors).

2. Prove that a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d > n$ is *not* rational (i.e. is not birationally isomorphic to $\mathbb{P}^{n-1}$).

3. Let $A \subset B$ be noetherian rings without zero divisors such that B is a finitely generated A-algebra. Let $0 \neq b \in B$. Prove that there is $0 \neq a \in A$ such that any homomorphism $\psi\colon A \to K$ (an algebraically closed field) with $\psi(a) \neq 0$ can be extended to a homomorphism $\tilde{\psi}\colon B \to K$ with $\tilde{\psi}(b) \neq 0$.

4. Prove that for a dominant finite type morphism $\pi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ between integral affine noetherian schemes, the image $\pi(\mathrm{Spec}(B))$ contains a dense open subset $U \subset \mathrm{Spec}(A)$.

5. A *constructible subset* of a scheme is a finite union of locally closed subsets. For example, the image of $\pi\colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$ is a constructible subset of $\mathbb{A}^2$, but not a locally closed one. Let $\pi\colon X \to Y$ be a finite type morphism of noetherian schemes, and let $S \subset X$ be a constructible subset. Prove that $\pi(S)$ is a constructible subset of Y (*Chevalley theorem*).