SET OF PROBLEMS 1.
5.10.2022

1. Conclude the proof of equivalence of the two definitions of Markov chains started at the lecture. Namely, show that Definition 1 of a Markov chain implies Definition 2.

2. Assume that a sequence of random variables $\xi_0, \xi_1, \ldots$ forms a Markov chain with a state space X. Prove that for any $n \geq 1$, $T > n$, sets $A \subset \underbrace{X \times \ldots \times X}_{T-n}$, $C \subset \underbrace{X \times \ldots \times X}_{n-1}$ and any $a \in X$,

$$\mathbb{P}\big((\xi_T, \ldots, \xi_{n+1}) \in A | \xi_n = a, (\xi_{n-1}, \ldots, \xi_0) \in C\big) = \mathbb{P}\big((\xi_T, \ldots, \xi_{n+1}) \in A | \xi_n = a).$$

   In particular, $\mathbb{P}(\xi_{n+k} = i | \xi_n = j, (\xi_{n-1}, \ldots, \xi_0) \in C) = \mathbb{P}(\xi_{n+k} = i | \xi_n = j)$.

3. Let a sequence of random variables $\xi_0, \xi_1, \ldots$ form a Markov chain with a state space X and $f : X \mapsto X$ be an injection. Is it true that the sequence $f(\xi_0), f(\xi_1), \ldots$ also must form a Markov chain? Is this true if f is not injective? It not, give a counterexample.

4. It is known that 80% of people who keep their savings in Sberbank continue to do so during the next year but the remaining 20% take off their money from Sberbank and invest them to the company "Roga and Kopyta"Ltd. Moreover, 40% of investors of this perspective company continue to invest there during the next year as well while a half of the remaining 60% put their money to Sberbank and a half invest them to Tesla. Finally, 70% of investors of Tesla continue to invest there during the next year, 20% put their money to Sberbank and 10% invest them to Roga and Kopyta.
   1) Find a Markov chain describing this process.
   2) Find a probability that a person who keeps his savings in Sberbank will continue to do so in 2 years.

5. Let a sequence of random variables $\xi_0, \ldots, \xi_T$ form a Markov Chain. Is it always true that the sequence $\xi_T, \ldots, \xi_0$ also does? If the answer is "yes" — prove it, otherwise give a counterexample.