Set of problems 2

7.10.2022

1. Prove that the following assertions are equivalent:

   a) Matrix A is stochastic.

   b) $Af^t \geq 0$ for any non-negative (string) vectors f, where t stands for the transposition, and, at the same time, $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)^t$.

   c) If a (string) vector μ is a distribution, then μA also is.

2. Prove that a product of two stochastic matrices is again a stochastic matrix.

3. Let $\xi_0, \xi_1, \ldots$ be a homogeneous Markov chain with the state space $\{1, 2, 3\}$, transition probability matrix $\Pi = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 2/3 & 0 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}$ and initial distribution $p^{(0)} = (1/3, 1/6, 1/2)$. Draw the corresponding graph. Find a) $p^{(2)}$ b) $\mathbb{P}(\xi_1 = 3, \xi_3 = 2)$.

4. Metro in the city St. Markovbourg is very strange: it consists of $m + 1$ stations, one of which is called Central station. There is a line connecting every station with the Central station, but there are no ways (except through the Central station) connecting any two other stations. High school student Vasya skips his classes by entering the metro at the Central station, taking train to any other station (all of them can be selected with the same probability), walking there and returning home late in the evening. In the morning he repeats this adventure. The choice of the destination station does not depend on which stations Vasya visited in the previous days. Denote by $\{\xi_n\}$ the number of stations visited by Vasya after n days, starting from zero's day (with the exception of the Central station): $\xi_0 = 1$, ξ_1 may be 1 or 2, ξ_2 may be $1, 2$ or 3, etc.

   - Explain why $\{\xi_n\}$ is a Markov chain on the appropriate state space X and the find transition probabilities of $\{\xi_n\}$.
   - Let τ_m be the first time Vasya has visited all stations, i.e. $\tau_m = \min\{n : \xi_n = m\}$. What is the distribution of τ_m for $m = 1$ and $m = 2$?
   - Compute the expectation $\mathbb{E}[\tau_m]$ for general $m \in \mathbb{N}$.