Set of problems 4

21.10.2022

1. Complete the proof of the theorem on the extinction probability for the Galton-Watson process: on the lecture we have not considered the case when $\mathbb{E}\eta_i^n > 1$.

2. Consider the following transition probability matrices:

$$a)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{pmatrix}, \quad c)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad d)\ \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Which of them are mixing?

   Hint: do not take the matrix to a power, look at the graph!

3. You have 4 pairs of gloves and (only one) office. Every morning you go from your home to the office and in the evening you go back. If it is cold in the morning then you take with you one pair of gloves. Otherwise, if it is not so cold then you do not take the gloves. You behave in the same way when leaving your office in the evening. If it is cold, but in the place where you are, there are no more gloves then you become cold. Initially you have 2 pairs of gloves at home and 2 pairs at the office. Find the averaged number of trips when you become cold for the first time (every day you make 2 trips — from home to the office and from the office to home).

   Hint: 1. Construct the Markov chain $\xi_0, \xi_1, \ldots$, where the state space consists of possible numbers of pairs of gloves in the place where you are now (so, the state space is $\{0, 1, \ldots, 4\}$).
   2. The average of course means the expectation. To find the averaged number of trips when you become cold for the first time, in particular you will have to find $\mathbb{E}\tau$, where τ is the number of trips you need to arrive to the state 0 for the first time.
   3. Study the conditional expectations $\mathbb{E}_a\tau$, $a = 0, \ldots, 4$ which are expectations of τ assuming that $\xi_0 = a$. Write down the system of equations connecting $\mathbb{E}_a\tau$ for different a and find its solution.

   Some theory on expectations and the conditional expectations as in item 3 above. Consider a probability space $(\Omega, \mathbb{P})$ and a random variable $\eta : \Omega \mapsto \mathbb{R}$ (as usual, discrete) satisfying $\mathbb{E}|\eta| < \infty$.

   a. (Law of total probability for expectations). For any sets $B_1, \ldots, B_n \subset \Omega$, $B_i \cap B_j = \emptyset$, satisfying $\cup_{i=1}^n B_i = \Omega$ and $\mathbb{P}(B_j) \neq 0$,

$$\mathbb{E}(\eta) = \sum_{j=1}^{n} \mathbb{E}(\eta|B_j)\mathbb{P}(B_j).$$

   Prove it! This is a simple exercise.

   b. For an event $A \subset \Omega$ we denote by $\mathbb{E}(\eta|A)$ the conditional expectation of η given A, that is the expectation of η with respect to the conditional probability $\mathbb{P}(\cdot|A)$:

$$\mathbb{E}(\eta|A) = \sum_{\omega \in \Omega} \eta(\omega)\mathbb{P}(\omega|A) = \sum_{c \in \eta(\Omega)} c\,\mathbb{P}(\eta = c|A).$$

Thus, the conditional expectation satisfies the same properties as the usual expectation (since we just replace one probability measure on Ω by another). In particular, it satisfies the law of total probability above.

c. What was told above naturally leads to the following question: for events $A, B \subset \Omega$ what is the conditional probability $\mathbb{Q}(\cdot|B)$, where $\mathbb{Q}(\cdot) := \mathbb{P}(\cdot|A)$? It is straightforward to see that

$$\mathbb{Q}(\cdot|B) = \mathbb{P}(\cdot|A \cap B).$$

Check this!

d. The expectations $\mathbb{E}_a\tau$ from item 3 of *Hint* above by their definition are given by $\mathbb{E}(\tau|\xi_0 = a)$. There I suggest you to write the law of total probability conditioning on the events of the form $\{\xi_1 = b\}$, $b = 0, \dots, 4$. Then you will get the expectations of the form $\mathbb{E}_a(\tau|\xi_1 = b) = \mathbb{E}(\tau|\xi_0 = a, \xi_1 = b)$, according to item c. "Clearly" , if $\mathbb{P}(\xi_0 = a, \xi_1 = b) \neq 0$, the latter expectation equals $\mathbb{E}(\tau|\xi_1 = b) = \mathbb{E}(\tau + 1|\xi_0 = b) = \mathbb{E}_b(\tau + 1)$, where the last equalities hold due to homogeneity of the chain.

e. This "clearly" from item d is intuitively obvious but requires the proof. **Check this!** You will have to use the Markov property of the sequence $\xi_0, \xi_1, \dots$.

Suggestions of the form **prove it/check this** *should be considered as a part of your homework which as well can be suggested as an exercise for a control work.*