Set of problems 6
19.11.2022

At the lecture, we discussed the Law of large numbers (LLN), classical one and one for Markov chains. Below I formulate the *strong* classic LLN (which I also mentioned at the lecture).

Theorem. Let the random variables $\eta_0, \eta_1, \ldots$ be independent and have finite second moments, i.e. $\mathbb{E}\eta_j^2 < \infty$ (in particular, this implies $\mathbb{E}|\eta_j| < \infty$ – prove it!), and let their variances $\operatorname{Var}\eta_j$ be bounded uniformly in j. Then the random variable $S_n = \sum_{j=0}^{n-1} \eta_j$ satisfies

$$\mathbb{P}\Big(\omega \in \Omega : \frac{S_n(\omega) - \mathbb{E}S_n}{n} \to 0 \text{ when } n \to \infty\Big) = 1.$$

This convergence is called *convergence almost surely (or almost everywhere)*. One writes $\frac{S_n - \mathbb{E}S_n}{n} \to 0$ a.s. (or a.e.). The almost sure convergence *implies* the convergence in probability, but the converse is not true. The type of convergence is the only difference between the strong and the standard LLN. The proof of the strong LLN is more complicated than that of the standard one, see for example Shiryaev's textbook.

For Markov chains, the strong LLN is also satisfied. Its formulation coincides with the formulation of the standard LLN given at the lecture, but instead of the convergence in probability we have the a.s. convergence. Below by writing LLN I mean the standard LLN. However, in some of the problems it is convenient to use the strong LLN; this is explicitly mentined in the comments.

1. Give an example of dependent but uncorrelated random variables.

2. Assume that two sequences of real valued random variables (ξ_n) and (η_n), defined on the same probability space, satisfy $\xi_n \to \xi$ and $\eta_n \to \eta$, in probability, for some random variables ξ and η. Prove that $\xi_n + \eta_n \to \xi + \eta$ in probability.

3. Assume that the real valued random variables $\xi_0, \xi_1, \ldots$ are iid (independent and identically distributed) and take only finite number of values. Is the classical LLN for them implied by the LLN for Markov chains?

4. Give an example of a (homogeneous) Markov chain $\xi_0, \xi_1, \xi_2, \ldots$ which is not ergodic but satisfies the assertion of the LLN.

   (note that in the formulation of the LLN we required the Markov chain to be exponentially ergodic.)

   Hint: perhaps it's more convenient to look for an example for which the strong LLN holds (of course, the standard LLN then also holds).

5. Give an example of a Markov chain $\xi_0, \xi_1, \xi_2, \ldots$ for which the LLN is not satisfied. That is, there exists a function f for which the limit (in probability) of the sequence $\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_k)$ either does not exists, or depends on the initial condition, or has the form different from that claimed in the LLN.

6. The company Roga and Kopyta Ltd. has problems due to an economical crisis and that is why it pays dividends to its shareholders irregularly. If in the present quarter it does not pay dividends, then in the next quarter it will not pay dividends as well with probability of 0.6. But if it pays dividends in the present quarter then in the next one it will also pay with a probability of 0.9. Assume that the company will not recover from the crisis for a sufficiently long time. Approximately which percentage of the maximum possible number of dividend payments over this period should its shareholders expect to receive?