Set of problems 8

02.12.2022

1. Consider a homogeneous Markov chain with transition probabilities (p_{ij}). Prove that if a distribution π satisfies the relation

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \forall i, j,$$

then π is a stationary state of this Markov chain. Is the inverse assertion true?

2. Using the Metropolis-Hastings algorithm, construct an ergodic Markov chain with three states such that the distribution $\pi = (7/15, 1/5, 1/3)$ is its (only) stationary state.

3. Let S_n be a set of all permutations of length $n \geq 1$. Given $\sigma \in S_n$ we take a random (not identical) transposition t and construct a new permutation $\sigma' = \sigma \circ t$ (all transpositions have equal probabilities). Then we repeat the procedure with the transposition σ', etc. Prove that the corresponding to this process Markov chain is ergodic and find its transition probabilities.

4. * Two gamblers flip a coin with probability $0 < p < 1$ of head and $q = 1 - p$ of tail. The first gambler has $A \in \mathbb{N}$ roubles and the second one has $B \in \mathbb{N}$ roubles. If the it is head then the second gambler gives one rouble to the first and if it is tail then the first gambler gives one rouble to the second. Then they flip the coin again and again. If a gambler does not have money any more then he loses. Find the probability that the first gambler will lose.

   Hint. This is a well-known "Gambler's ruin problem" written in many textbooks. See e.g. "Probability" by Shiryaev.