Set of problems 7
25.11.2022

Consider a homogeneous Markov chain $\xi_0, \xi_1, \dots$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a (finite or countable) set of states X (so that $\xi_j : \Omega \mapsto X$). Consider a random variable τ on the same probability space such that for every $n \geq 0$ the event $\{\tau = n\}$ belongs *to the algebra generated by the random variables* $\xi_0, \dots, \xi_n$. This means that for every n there is a set $A_n \subset X^n$ [1], such that that the set $\{\omega \in \Omega : \tau(\omega) = n\}$ has the form $\{\omega \in \Omega : (\xi_0, \dots, \xi_n)(\omega) \in A_n\}$. Such a random variable τ is called *stopping time*. Informally speaking, τ is a stopping time if for each n we can determine whether the event $\{\tau = n\}$ has occurred knowing the trajectory of the Markov chain up to the time n. For example, the random variable $\tau_A = \inf\{n \geq 0 : \xi_n \in A\}$, where $A \subset X$, is a stopping time (its meaning is the first time when the process enters to the set A; here and below $\inf \emptyset := \infty$). But the random variable $\widetilde{\tau}_A = \inf\{n \geq 0 : \xi_{n+1} \in A\}$ is not a stopping time. Note that any constant is also a stopping time.

1. Consider a homogeneous Markov chain $\xi_0, \xi_1, \dots$ with transition probabilities (p_{ij}) and a stopping time τ. Prove the *strong Markov property*:

$$\mathbb{P}(\xi_{\tau+1} = j | \xi_\tau = i, (\xi_{\tau-1}, \dots, \xi_0) \in B_{<\tau}, \tau < \infty) = \mathbb{P}(\xi_{\tau+1} = j | \xi_\tau = i, \tau < \infty) = p_{ij},$$

   for all i, j and an arbitrary collection of sets $B_{<n} \subset X^{\times n}$, $n \geq 1$.

   Comment: If $\tau = \mathit{const}$, then the strong Markov property coincides with the usual one. It can be shown that the strong Markov property can be also stated as follows: given that $\tau < \infty$ and $\xi_\tau = i$, the random process $(\xi_{\tau+n})_{n\geq 1}$ does not depend on the process $(\xi_n)_{n \leq \tau}$ and has the same distribution as the original process $(\xi_n)_{n\geq 0}$, taken under the condition that $\xi_0 = i$. Now it is clear why the strong Markov property is so often used in solving various problems related to Markov chains. In particular, it allows to describe behaviour of the chain after it enters some set $A \subset X$, that is, starting from the moment τ_A. Indeed, it behaves in the same way as the original chain but with the initial condition at the point through which it entered A.

2. * Consider an exponentially ergodic Markov chain with the state space $X = \{1, 2, \dots\}$ and the stationary distribution π. Assume that $\pi_1 > 0$. Consider the following sequence of stopping times:

$$\tau_1 = \{\inf k > 0 : \xi_k = 1\}, \quad \tau_n = \{\inf k > \tau_{n-1} : \xi_k = 1\}, \quad n \geq 2,$$

   where $\inf \emptyset := \infty$. Thus, τ_n— n-th moment when the process enters the state 1.

   a) Prove that for every initial distribution $p^{(0)}$ we have $\mathbb{E}(\tau_1)^r < \infty$ for any $r \geq 0$ (we say that the random variable τ_1 has *finite moments*). As a corollary, show that for each initial distribution $p^{(0)}$ we have $\mathbb{P}(\tau_1 < \infty) = 1$.

   Hint: using the convergence of transition probabilities in n steps $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$, which follows from the ergodicity of the chain, estimate from above the probability $\mathbb{P}(\tau_1 > k)$.

[1] This is the direct product of n copies of the set X.

b) Prove that the random variables τ_1 and $\tau_2 - \tau_1$ are independent, and if the initial distribution satisfies $p_1^{(0)} = 1$ (that is, at the initial time we are at the state 1), then their distributions coincide. Show that this implies, in particular, $\mathbb{E}(\tau_2)^r < \infty \ \forall r > 0$.

c) Arguing similarly, prove that $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$ is a sequence of independent random variables. Show that these random variables, except τ_1, have the same distribution, and in the case when $p_1^{(0)} = 1$, τ_1 also has the same distribution. Prove that $\mathbb{E}(\tau_n)^r < \infty \ \forall r > 0$.

d) Prove that $\tau_n/n \to \mathbb{E}(\tau_2 - \tau_1)$ for $n \to \infty$, a.s.

Hint: let $\nu_1(n) = \#\{0 \le i \le n-1 : \xi_i = 1\}$. *Using the strong LLN for Markov chains, find the limit of the sequence* $\nu_1(\tau_n)/\tau_n$. *What is* $\nu_1(\tau_n)$ *equal to?*

e) Prove that $\mathbb{E}(\tau_2 - \tau_1) = (\pi_1)^{-1}$.

We have got the following non-obvious fact: the average time between the first and second visits of a given state equals to the inverse of the stationary probability of this state. Qualitatively, this result is intuitive, but quantitatively not at all, I think.