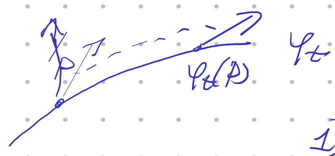
Производная Ли X , Y – вект. поля

$\swarrow$ φ_t



$$L_X(Y) = \frac{d}{dt} d\varphi_t^{-1}(Y(\varphi_t(p)))\Big|_{t=0}$$

1) $L_X(Y) = [X,Y]$

2) $L_X(T_1 \otimes T_2) = L_X(T_1)\otimes T_2 + T_1 \otimes L_X(T_2)$

$A \otimes B$ · комп-ты перемн. а матр. индексы i j

$$L_X(\omega) = \frac{d}{dt} d\varphi_t^{*}(\omega(\varphi_t(p)))\Big|_{t=0}$$

3) L_X (свертка) = свертка(L)

$\Downarrow$

Y – вект. поле ω – 1 форма

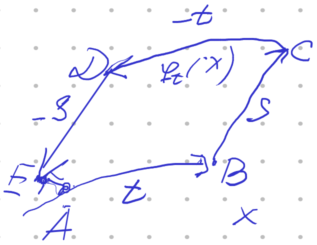
$$X((\omega,Y)) = L_X((\omega,Y)) = (L_X(\omega),Y) + (\omega, L_X(Y))$$

(ω,Y) $\omega = \omega_i dx^i$ $Y = B^j \frac{\partial}{\partial x^j}$

$$(\omega,Y) = \sum_i \omega_i B^i = (\omega \otimes Y)^i_i$$

свертка тенз. пр-ния

Геом. смысл к-та вект. полей



$C = \varphi_s(Y)(B)$

$B = \varphi_t(X)(A)$

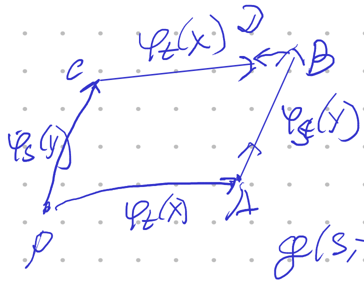
$\varphi_t(X)$ – поток

$X = \frac{\partial}{\partial x}$ $Y = \frac{\partial}{\partial y}$

$\varphi_{-s}(Y)\varphi_{-t}(X)\varphi_s(Y)\varphi_t(X)$

$\overrightarrow{AE} \approx ts[X,Y]$

∂З $\varphi_t(x)($



$\varphi_s(y)\circ\varphi_t(x) - \varphi_t(x)\circ\varphi_s(y)\ (P)$

f – ф-ция на M

$g(s,t) = f(\varphi_s(y)\varphi_t(x) - \varphi_t(x)\varphi_s(y)(P))$

$g(s,t) = 0 + st\cdot ? + o(s^2+t^2)$ $\quad \frac{\partial g}{\partial s}\Big|_{t=0}$ $\quad \varphi_t(0)=Id$, $\varphi_s(0)=Id$

$f = x^i$

1) $x^i(A) = x^i(P) + ta^i(0) + o(t^2)$ $\quad X = \sum a^i \frac{\partial}{\partial x^i}$

$P=0$

2) $x^i(B) = x^i(A) + sb^i(A) + o(s^2) =$ $\quad Y=\sum b^i\frac{\partial}{\partial x^i}$

$= x^i(0)+ta^i(0)+s\left(b^i(0)+t\frac{\partial b^i}{\partial x^j}a^j(0)\right) + o(s^2+t^2)$

3) $x^i(C) = x^i(0)+sb^i(0)+o(s^2)$

4) $x^i(D) = x^i(C)+ta^i(C)+o(t^2) = x^i(0)+sb^i(0) +$

$+t\left(a^i(0)+\frac{\partial a^i}{\partial x^j}b^j(0)s\right) + o(s^2+t^2)$

$x^i(B)-x^i(D) = st\left(a^j\frac{\partial b^i}{\partial x^j} - b^j\frac{\partial a^i}{\partial x^j}\right)(0)$

$\to$ i-я комп. $[X,Y]$

$T^{i_1\dots i_p}_{j_1\dots j_q}$ $\quad \left(\sum_{i_p} T^{i_1\dots i_p}_{j_1\dots j_{q-1} i_p}\right)$ $\quad (u,v) = (u\otimes v)$ свертка

$[X,Y]=\sum c^i\frac{\partial}{\partial x^i}$ $\quad x^i(r) = x^i(0)+c^i\cdot r$

Дифференциальные формы

Внешние формы $\quad$ грассманова алгебра

V $\quad V^*$ $\quad \Lambda^k(V^*)$ – кососим. k-линейные формы на V

$\dim\Lambda^k(V^*) = \binom{n}{k}$ $\quad \Lambda^k(V^*)\subset \underbrace{V^*\otimes\dots\otimes V^*}_{k}$

Базис $e_1,\dots,e_n$ в V $\quad e^1,\dots,e^n$ – базис V^* двойств.

$(e^i,e_j)=\delta^i_j$

Базис в $\Lambda^k(V^*)$ $\quad i_1<\dots<i_k$

$e^{i_1}\wedge e^{i_2}\wedge\dots\wedge e^{i_k} = \sum_{\sigma\in S_k}(-1)^\sigma e^{i_{\sigma 1}}\otimes e^{i_{\sigma 2}}\dots\otimes e^{i_{\sigma k}}$

$\Lambda^*(V) = \oplus\Lambda^k(V^*)$ образуют грассм. алгебру отн-но $\wedge$

1) $(e^{i_1}\wedge\dots\wedge e^{i_k})\wedge(e^{j_1}\wedge\dots\wedge e^{j_l}) = e^{i_1}\wedge\dots\wedge e^{i_k}\wedge e^{j_1}\wedge\dots\wedge e^{j_l}$

2) $\omega^k \in \Lambda^k(V^*)$ $\quad \omega^e \in \Lambda^e(V^*)$

$$(\omega^k \wedge \omega^e)(v_1, v_2, \dots v_{k+e}) = \sum_{\substack{\sigma \in S_{k+e} \\ \sigma_1 < \dots < \sigma_k \\ \sigma_{k+1} < \dots < \sigma_{k+e}}} \omega^k(v_{\sigma_1}, \dots v_{\sigma_k}) \cdot \omega^e(v_{\sigma_{k+1}}, \dots v_{\sigma_{k+e}})$$

$\Lambda^*(V)$ – алгебра ассоц. и суперком-на

$$\omega^k \wedge \omega^e = (-1)^{ke} \omega^e \wedge \omega^k$$

Базис $e^{i_1} \wedge e^{i_2} \dots \wedge e^{i_k}$

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}$$

$\leftarrow$ координаты формы

Смена базиса. V $\{e_i\} \to \{f_i\}$ $\quad e_i = A_i^j f_j$

$\{e^i\} \to \{f^j\}$ $\quad e^i = B^i_j f^j$

B – обр. матрица к A

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{i_1 \dots i_k} \omega_{i_1 \dots i_k} e^{i_1} \otimes \dots \otimes e^{i_k} =$$

$\omega_{i_1 \dots i_k} = \omega_{i_{\sigma_1} \dots i_{\sigma_k}} \cdot (-1)^\sigma$

$$= \sum_{i_1 \dots i_k} \omega_{i_1 \dots i_k} B^{i_1}_{j_1} \dots B^{i_k}_{j_k} f^{j_1} \otimes \dots \otimes f^{j_k} =$$

$$= \sum_{i_1 < \dots i_k} \omega_{i_1 \dots i_k} B^{[i_1 \dots i_k]}_{[j_1 \dots j_k]} f^{j_1} \wedge \dots \wedge f^{j_k}$$

$\leftarrow$ минор матрицы B со стр-ками $i_1 \dots i_k$ столбцами $j_1 \dots j_k$

$$\omega'_{j_1 \dots j_k} = \sum_{i_1 < \dots < i_k} B^{[i_1 \dots i_k]}_{[j_1 \dots j_k]} \omega_{i_1 \dots i_k}$$

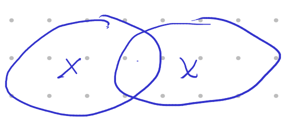
$V \to TM$ – касат. рассл

$V^* — T^*M$ – кокасат. рассл

$\Lambda^k(V^*) \to \Lambda^k T^*M$

TM – касат. рассл. $\quad g_{\alpha\beta} = J_{\alpha\beta}$ – матр. Якоби

T^*M $\quad g_{\alpha\beta} = (J_{\alpha\beta}^{-1})^t$ $\quad : \mathbb{R}^{n*} = (\mathbb{R}^n)^*$



$(T^*M)^{\otimes k}$ $\quad g_{\alpha\beta} = (J_{\alpha\beta}^{-1})^{t \otimes k}$ $\quad : (\mathbb{R}^{n*})^{\otimes k} \to (\ )^{\otimes k}$

$\shortparallel$ $A^{\otimes k}$

$(\mathbb{R}^{n*})^{\otimes k} \supset \Lambda^k(\mathbb{R}^{n*})$ — кососим. формы

$(J_{\alpha\beta}^{-1})^t = A$ $\quad A^{\otimes k}: \Lambda^k(\mathbb{R}^{n*}) \hookrightarrow \Lambda^k(\mathbb{R}^{n*})$

$$A \otimes A(v \otimes w - w \otimes v) = $$
$$= Av \otimes Aw - Aw \otimes Av$$

$$\left.\begin{matrix} \varphi_{\alpha\beta} \\ \| \\ A^{\otimes k} \end{matrix}\right|_{\Lambda^k(\mathbb{R}^{n*})}$$

Коцикл в $\Lambda^k(\mathbb{R}^{n*})$ — это ограничение $(J_{\alpha\beta}^{-1\,t})^{\otimes k}$ на косые формы.

k-форма $\omega^k(\bar{x})$ — сечение $\Lambda^k(T^*M)$

пр-во всех k-форм обозн. $\Omega^k(M)$

$x \in M \Rightarrow \omega^k(x) \in \Lambda^k(T^*_x M)$

Карта $(U_\alpha, \bar{x})$ $\quad V \Rightarrow TM \quad V^* \Rightarrow T^*M$

базис сечений TM $\quad \{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ $\quad e_1 \dots e_n$

базис сечений T^*M $\quad \{dx^1, \dots dx^n\}$

Базис сечений $\Lambda^k T^* M$

$dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad i_1 < \dots < i_k$

Произв. k-форма

$$\omega^k = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(x)\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

Замена коорд. y

$$\omega_{i'_1 \dots i'_k}(y) = \frac{\partial x^{[i_1 \dots i_k]}}{\partial y^{[i'_1 \dots i'_k]}}\, \omega_{i_1 \dots i_k}(x)$$

$n = 2$

$$\sum_{i_1 < i_2} \omega_{i_1 i_2}\, dx^{i_1} \wedge dx^{i_2} = \sum_{i_1 < i_2} \omega_{i_1 i_2} (dx^{i_1} \otimes dx^{i_2} - dx^{i_2} \otimes dx^{i_1}) =$$

$$= \sum_{j_1, j_2} \sum_{i_1 < i_2} \left(\frac{\partial x^{i_1}}{\partial y^{j_1}} \frac{\partial x^{i_2}}{\partial y^{j_2}} dy^{j_1} \otimes dy^{j_2} - \frac{\partial x^{i_2}}{\partial y^{j_1}} \cdot \frac{\partial x^{i_1}}{\partial y^{j_2}} dy^{j_1} \otimes dy^{j_2} \right) \omega_{i_1 i_2}$$

$$= \sum_{j_1 j_2} \sum_{i_1 < i_2} \omega_{i_1 i_2} \left| \frac{\partial x^{i_1}}{\partial y^{j_1}} \frac{\partial x^{i_2}}{\partial y^{j_2}} \right| dy^{j_1} \otimes dy^{j_2} = \sum_{j_1 < j_2} \sum_{i_1 < i_2} \omega_{i_1 i_2} \left| \frac{\partial x^{i_1}}{\partial y^{j_1}} \frac{\partial x^{i_2}}{\partial y^{j_2}} \right| dy^{j_1} \wedge dy^{j_2}$$

$$\left| \frac{\partial x}{\partial y} \right|^{i_1 i_2}_{j_1 j_2} \qquad \left| \frac{\partial x}{\partial y} \right|^{i_1 i_2}_{j_1 j_2}$$

Имеется вн. умн-ние $\Omega^k(M)\otimes\Omega^e(M)\to\Omega^{k+e}(M)$

$$\omega^k\wedge\omega^e(X_1,\dots X_{k+e})=\sum_{\substack{b\in S_{k+e}\\ b_1<\dots<b_k\\ b_{k+1}<\dots<b_{k+e}}}\omega^k(X_{b_1},\dots X_{b_k})\cdot\omega^e(X_{b_{k+1}},\dots X_{b_{k+e}})$$

X_1 — вект. поля

$\omega^k\in\Omega^k(M)$ $\qquad$ $\omega^k(X_1,\dots X_k)\in C^\infty(M)$

X — вект. поле, подстановка вект. поля

i_X — иот $\qquad$ $i_X(\omega^k)$ — $(k-1)$ форма

$$i_X(\omega^k)(X_1,\dots X_{k-1})=\omega^k(X,X_1\dots X_{k-1})$$

<u>пример</u> $\quad \omega=dx\wedge dy \qquad X=y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$

$i_X(\omega)$ — 1 форма

$$i_X(\omega)\left(\frac{\partial}{\partial x}\right)=\omega\left(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right)=(dx\otimes dy-dy\otimes dx)\left(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right)=-x$$

$$i_X(\omega)\left(\frac{\partial}{\partial y}\right)=y \qquad i_X(\omega)=-x\,dx+y\,dy.$$