Листок 2

Срок сдачи: 16 мая.

Problem 1 *Let A, B_1 and B_2 be three elements of an associative algebra. Let*

$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \quad \text{and we understand } A \text{ as } \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Parameters a_1, a_2 and a_3 are in $\mathbb{C}$ and we denote

$$\sigma = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us introduce dependence on m_i by means of relations

$$F^{(1)}(m) = F(m_1 + 1, m_2, m_3), \qquad F^{(2)}(m) = F(m_1, m_2 + 1, m_3),$$
$$F^{(2,3)}(m) = F(m_1, m_2 + 1, m_3 + 1), \quad \text{etc.}$$

and

$$B^{(1)} = (A - a_1)B(A - a_1)^{-1}, \qquad B^{(2)} = (A - a_2)B(A - a_2)^{-1},$$
$$B^{(3)} = (A - a_3\sigma)B(A - a_3\sigma)^{-1}.$$

Find commutator identity and corresponding evolution equation.

Problem 2 *Under the same conditions as in Problem 1 but in the case of evolution*

$$B^{(1)} = (A - a_1)B(A - a_1)^{-1}, \qquad B^{(2)} = (A - a_2\sigma)B(A - a_2\sigma)^{-1},$$
$$B^{(3)} = (A - a_3\sigma)B(A - a_3\sigma)^{-1},$$

find commutator identity and corresponding evolution equation.

Problem 3 *Under the same conditions as in Problem 1 but in the case of evolution*

$$B^{(1)} = (A - a_1\sigma)B(A - a_1\sigma)^{-1}, \qquad B^{(2)} = (A - a_2\sigma)B(A - a_2\sigma)^{-1},$$
$$B^{(3)} = (A - a_3\sigma)B(A - a_3\sigma)^{-1},$$

find commutator identity and corresponding evolution equation.

Problem 4 *Let F be a diagonal matrix, i.e., its symbol is z-independent: $\widetilde{F}(m_1, z) \equiv \widetilde{F}(m_1)$. Prove that under this condition equality*

$$\widetilde{FG}(m_1, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(m_1, z\zeta) \sum_{m'_1 \in \mathbb{Z}} \zeta^{m_1 - m'_1} \widetilde{G}(m'_1, z)$$

takes the form

$$\widetilde{FG}(m_1, z) = \widetilde{F}(m_1)\widetilde{G}(m_1, z)$$

for any operator G.

Problem 5 *Let G be a function of the shift operator only, i.e., its symbol is independent of the discrete variable, $\widetilde{G}(m_1, z) \equiv \widetilde{G}(z)$. Prove that under this condition equality*

$$\widetilde{FG}(m_1, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(m_1, z\zeta) \sum_{m'_1 \in \mathbb{Z}} \zeta^{m_1 - m'_1} \widetilde{G}(m'_1, z)$$

takes the form

$$\widetilde{FG}(m_1, z) = \widetilde{F}(m_1, z)\widetilde{G}(z)$$

for an arbitrary operator F.

Problem 6 *Proof that relations*

$$p_1(T) = T, \qquad p_2(T) = T + a_{12}, \qquad p_3(T) = (T + a_1)^2 - a_3^2,$$

give equation

$$\begin{aligned} &\big[(\Delta_1 a_1 - \Delta_2 a_2)^2 - a_3^2(\Delta_1 - \Delta_2)^2\big]\Delta_3 B = \\ &\qquad = a_{12}\Delta_1\Delta_2\big(a_{12}\Delta_1\Delta_2 + 2\Delta_1 a_1 - 2\Delta_2 a_2\big)B. \end{aligned}$$