Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 17.01.2023

These exercises are due by January 24th. This is a general rule: the due date is one week after the assignment. The final grade for the course is calculated as 0.1 of the percentage of completely solved exercises. You may submit e.g. the high quality scans of your handwritten solutions in the natural order. I will grade neither poor quality scans nor randomly ordered scans. You may also submit your handwritten solutions as a hardcopy or solutions typeset in TeX.

1. Construct isomorphisms a) $\mathrm{SL}(2, \mathbb{F}_2) \simeq \mathfrak{S}_3$.
b) $\mathrm{PGL}(2, \mathbb{F}_3) \simeq \mathfrak{S}_4$.
c) $\mathrm{PSL}(2, \mathbb{F}_4) \simeq \mathfrak{A}_5 \simeq \mathrm{PSL}(2, \mathbb{F}_5)$.
d) $\mathrm{PSL}(2, \mathbb{F}_9) \simeq \mathfrak{A}_6$.

2. Construct an isomorphism $\mathrm{PSL}(2, \mathbb{Z}/4\mathbb{Z}) \simeq \mathfrak{S}_4$.

3. Prove that for q odd, $\mathrm{SL}(2, \mathbb{F}_q)$ has two conjugacy classes of cardinality 1, four conjugacy classes of cardinality $\frac{q^2-1}{2}$, and $\frac{q-3}{2}$ classes of cardinality $q(q+1)$, and $\frac{q-1}{2}$ conjugacy classes of cardinality $q(q-1)$.

4. Count the conjugacy classes of $\mathrm{SL}(2, \mathbb{F}_q)$ for $q = 2^n$.

5. Given subgroups $H \subset G \supset J$ and their representations (V, ρ) and (W, σ), prove that the assignment $F \mapsto T_F$ defines an isomorphism from $\mathcal{F} := \{F \colon G \to \mathrm{Hom}_{\mathbb{C}}(V, W) \mid F(jgh) = \sigma(j)F(g)\rho(h)\}$ onto $\mathrm{Hom}_G\left(\mathrm{Ind}_H^G(V), \mathrm{Ind}_J^G(W)\right)$, where $(T_F f)(g) := \frac{1}{|G|:|H|} \sum_{r \in G} F(gr^{-1})(f(r))$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 24.01.2023

1. Prove that a) the group $P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},\ a \in \mathbb{F}_q^\times,\ b \in \mathbb{F}_q \right\}$ is isomorphic to the group of affine transformations of the line $\mathbb{F}_q$.
b) Its irreducible $(q-1)$-dimensional representation is isomorphic to the natural representation in the space of functions on $\mathbb{F}_q$ with the zero sum.

2. Let $q = p^n$. Consider the additive character $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \ni k \mapsto \exp\left(\frac{2\pi\sqrt{-1}k}{p}\right) \in \mathbb{C}^\times$ and compose it with the homomorphism $\mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_p} \colon \mathbb{F}_q \to \mathbb{F}_p$ to obtain the additive character $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$. Prove that any character $\mathbb{F}_q \to \mathbb{C}^\times$ has a form $\psi_a(x) := \psi(ax)$ for some $a \in \mathbb{F}_q$.

3. Define the *Fourier transform* $\mathrm{FT} \colon \mathbb{C}[\mathbb{F}_q] \to \mathbb{C}[\mathbb{F}_q]$ by the formula

$$\mathrm{FT}(f)(b) := \sqrt{q}^{-1} \sum_{a \in \mathbb{F}_q} f(a)\psi(ab).$$

Prove that $\mathrm{FT}(\mathrm{FT}(f))(a) = f(-a)$ (Plancherel formula).

4. Prove that $\sum_{a\in\mathbb{F}_q}|f(a)|^2=\sum_{b\in\mathbb{F}_q}|\mathrm{FT}(f)(b)|^2$ (Parseval identity).

5. For $b\in\mathbb{F}_q$ let $T_b\colon \mathbb{C}[\mathbb{F}_q]\to\mathbb{C}[\mathbb{F}_q]$ be the translation operator $T_bf(a):=f(a+b)$. Prove that

a) $\mathrm{FT}\circ T_b=\psi_{-b}\cdot\mathrm{FT}$ (the composition of the Fourier transform with the operator of pointwise multiplication by the function ψ_b).

b) $\mathrm{FT}(\psi_b\cdot f)=T_b\circ\mathrm{FT}(f)$.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 31.01.2023

1. In the setup of problem 5 of January 17th, if $J=H$ and $(W,\sigma)=(V,\rho)$, prove that the algebra structure on $\mathrm{End}_G(\mathrm{Ind}_H^G V)$ (with respect to composition) corresponds (under the isomorphism of problem 5) to the convolution operation $(F_1{*}F_2)(g)=\frac{1}{|G|:|H|}\sum_{r\in G}F_1(gr^{-1})F_2(r)$.

2. Let $G=\mathrm{GL}(n,\mathbb{F}_q)$, and let $B\subset G$ be the Borel subgroup of upper-triangular matrices. Prove that $B\backslash G/B=\mathfrak{S}_n$ (the matrices of permutations are the representatives of double cosets). This is called the *Bruhat decomposition* (or *Gauß method*).

3. The algebra $\mathcal{H}_q:=\mathrm{End}_G(\mathrm{Ind}_B^G\mathbb{C})$ is called the *Iwahori-Hecke* algebra. It has a basis $\{T_w,\ w\in\mathfrak{S}_n\}$ of characteristic functions of the double cosets multiplied by $|G|:|B|^2$. Let $s_i=(i,i+1)\in\mathfrak{S}_n$ be a simple transposition. Prove that

a) $(T_{s_i}-q)(T_{s_i}+1)=0$.

b) $T_{s_i}T_{s_{i+1}}T_{s_i}=T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$ and $T_{s_i}T_{s_j}=T_{s_j}T_{s_i}$ if $|i-j|>1$ (*braid relations*).

c) $T_yT_w=T_{yw}$ if $\ell(yw)=\ell(y)+\ell(w)$, where $\ell(y)$ is the number of disorders in the permutation $y\in\mathfrak{S}_n$ (that is the length of a shortest word in the generators s_i representing y).

4. Prove that $\mathcal{H}_q$ is generated by $T_{s_1},\dots,T_{s_{n-1}}$ with relations 3(a,b).

5. Prove that the algebra $\mathcal{H}_q$ is semisimple (i.e. $\mathcal{H}_q$ is isomorphic to a direct sum of matrix algebras).

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 07.02.2023

1. Let $V=\Lambda\oplus\Lambda'$ be a symplectic vector space (over $\mathbb{F}_q$) decomposed into a direct sum of two Lagrangian subspaces (so that $\Lambda'\cong\Lambda^*$). Let us write $g\in\mathrm{GL}(V)$ in the block form according to this decomposition: $g=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$. In this problem set we follow the convention that g acts on V on the *right*: i.e. as on vector-rows. We set $g^I:=\begin{pmatrix}\delta^t&-\beta^t\\ -\gamma^t&\alpha^t\end{pmatrix}$ (the transposition denotes taking adjoint with respect to the identification $\Lambda'\cong\Lambda^*$). Prove that

a) $g\in\mathrm{Sp}(V)\subset\mathrm{GL}(V)$ iff $g\cdot g^I=\mathrm{Id}$.

b) The convolution algebra of complex functions on $\mathrm{Sp}(V)\backslash\mathrm{GL}(V)/\mathrm{Sp}(V)$ is commutative.

2. We introduce the following symplectic automorphisms. For $\alpha \in \mathrm{Aut}(\Lambda)$ we set $d_0(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^t)^{-1} \end{pmatrix}$. For a quadratic form Q on Λ we set $t_0(Q) := \begin{pmatrix} \mathrm{Id} & Q \\ 0 & \mathrm{Id} \end{pmatrix}$. For an isomorphism $\gamma\colon \Lambda' \xrightarrow{\sim} \Lambda$ we set $d_0'(\gamma) := \begin{pmatrix} 0 & -(\gamma^t)^{-1} \\ \gamma & 0 \end{pmatrix}$. Prove that if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}(V)$, and γ is invertible, then

$$g = \begin{pmatrix} \mathrm{Id} & \alpha\gamma^{-1} \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & -(\gamma^t)^{-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & \gamma^{-1}\delta \\ 0 & \mathrm{Id} \end{pmatrix}.$$

3. The symplectic group $\mathrm{Sp}(V)$ acts on the Heisenberg group $0 \to \mathbb{F}_q \to H \to V \to 0$, and (projectively) on the irreducible H-module $(W_\psi \simeq \mathbb{C}[\Lambda], \rho_\psi)$ with the central character $\psi\colon \rho_\psi(h)A_g = A_g\rho_\psi(h^g)$. Prove that we can take $(A_{d_0(\alpha)}f)(\lambda) = f(\lambda\alpha)$.

4. Prove that we can take $(A_{t_0(Q)}f)(\lambda) = \psi(\frac{Q(\lambda)}{2})f(\lambda)$.

5. Prove that we can take $(A_{d_0'(\gamma)}f)(\lambda) := \mathrm{FT}(f)(-\lambda(\gamma^t)^{-1})$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 14.02.2023

1. Let $\begin{pmatrix} 0 & -(\gamma_1^t)^{-1} \\ \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & Q \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & -(\gamma_2^t)^{-1} \\ \gamma_2 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & Q_1 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & -(\gamma^t)^{-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & Q_2 \\ 0 & \mathrm{Id} \end{pmatrix}$. Prove that $Q = \gamma_1^{-1}\gamma\gamma_2^{-1}$.

2. Prove that in the setup of Problem 1, we have $A_{d_0'(\gamma_1)}A_{t_0(Q)}A_{d_0'(\gamma_2)} = \Gamma(Q)A_{t_0(Q_1)}A_{d_0'(\gamma)}A_{t_0(Q_2)}$.

3. Let $u_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$, $u_1 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$, and $u_1u_2 = u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U$ (that is, $\gamma, \gamma_1, \gamma_2$ are all invertible). Prove that the cocycle $c(u_1, u_2) = \Gamma(\gamma_1^{-1}\gamma\gamma_2^{-1})$.

4. Prove that the Witt group $\mathrm{Witt}(\mathbb{F}_q)$ is $\mathbb{Z}/4\mathbb{Z}$ if $-1 \notin (\mathbb{F}_q^\times)^2$, and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ otherwise.

5. Prove that for $q \geq 5$, given any quadruple of Lagrangian subspaces in $\mathbb{F}_q^{2n}$, there is a fifth subspace transversal to any one of the quadruple.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 21.02.2023

1. Prove that the derived subgroup $[\mathrm{Sp}(V), \mathrm{Sp}(V)] = \mathrm{Sp}(V)$ for odd q.

2. Given $a, b \in \mathbb{F}_q^\times$, we consider the quaternion algebra $H_{a,b}$ over $\mathbb{F}_q$ with generators i, j and relations $i^2 = a,\ j^2 = b,\ ij = -ji$. Prove that $H_{a,b} \simeq \mathrm{Mat}_{2\times 2}(\mathbb{F}_q)$.

3. For an arbitrary ground field F in place of $\mathbb{F}_q$, prove that $H_{a,b} \simeq \mathrm{Mat}_{2\times 2}(F)$ iff the quadratic form $Q = x^2 - ay^2 - bz^2 + abw^2$ on F^4 represents 0 (i.e. the equation $Q = 0$ has a nontrivial solution), and otherwise $H_{a,b}$ is a skew-field (a division algebra).

4. For an $\mathbb{F}_q$-vector space $\Lambda = \mathbb{F}_q^n$, and $f \in \mathbb{C}[\Lambda]$ we define the Radon transform $\operatorname{RT} f \in \mathbb{C}[\Lambda^*]$ as $(\operatorname{RT} f)(\lambda^*) := q^{-n/2} \sum_{\lambda\in\Lambda\,:\,\langle\lambda,\lambda^*\rangle=1} f(\lambda)$. For a character $\theta\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ let $\mathbb{C}[\Lambda]_\theta$ be the set of functions such that $f(a\lambda) = \theta(a)f(\lambda)$. Prove that both RT and FT take $\mathbb{C}[\Lambda]_\theta$ to $\mathbb{C}[\Lambda^*]_{\theta^{-1}}$ and $\mathrm{FT} = \gamma_\theta \operatorname{RT}$ on $\mathbb{C}[\Lambda]_\theta$ for nontrivial θ, where $\gamma_\theta := \sum_{x\in\mathbb{F}_q^\times} \theta(x)\psi(x)$.

5. Prove that the Weil representations of $\mathrm{SL}(2,\mathbb{F}_q)$ in $\mathbb{C}[\Lambda]^\pm$ (for $\Lambda = \mathbb{F}_q$) for additive characters ψ, ψ' such that there is no $a\ :\ \psi'(x) = \psi(a^2x)$, are *not* isomorphic (so this way we obtain two different $\frac{q+1}{2}$-dimensional representations of $\mathrm{SL}(2,\mathbb{F}_q)$ and two different $\frac{q-1}{2}$-dimensional representations of $\mathrm{SL}(2,\mathbb{F}_q)$).

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 28.02.2023

1. Prove that a) $g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g^{-1} \in U$ iff $g \in B$.

b) The normalizer of B in G is B.

c) $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \in B$ iff $\gamma = 0$ or $\delta = 0$.

2. Given multiplicative characters $\mu_1 \neq \mu_2\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$, we denote by π_{μ_1,μ_2} the irreducible induced representation $\operatorname{Ind}_B^G(\mu_1,\mu_2)$. There are $\frac{(q-1)(q-2)}{2}$ pairwise non-isomorphic representations of this type. Prove that the character value of π_{μ_1,μ_2} at

a) $c_1(x) := \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ equals $(q+1)\mu_1(x)\mu_2(x)$.

b) $c_2(x) := \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ equals $\mu_1(x)\mu_2(x)$.

c) $c_3(x,y) := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ equals $\mu_1(x)\mu_2(y) + \mu_1(y)\mu_2(x)$.

d) $c_4(z)$ (a matrix conjugate to the one of multiplication by $z \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ acting on $\mathbb{F}_{q^2} = \mathbb{F}_q^2$) equals 0.

3. Given a multiplicative character $\mu\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$, we denote by St_μ the irreducible q-dimensional subrepresentation of $\operatorname{Ind}_B^G(\mu,\mu)$. There are $q-1$ pairwise non-isomorphic representations of this type. Prove that the character value of St_μ at

a) $c_1(x)$ equals $q\mu(x)^2$.

b) $c_2(x)$ equals 0.

c) $c_3(x,y)$ equals $\mu(xy)$.

d) $c_4(z)$ equals $-\mu(z\bar{z})$ (where $\bar{z} := z^q$).

4. Given a nontrivial additive character $\psi\colon U = \mathbb{F}_q \to \mathbb{C}^\times$ we denote by Whit_ψ the Whittaker model representation $\operatorname{Ind}_U^G\psi$. It has dimension $(q^2-1)(q-1)$. Prove that the character value of Whit_ψ (the Gelfand-Graev character) at

a) $c_1(x)$ equals $(q^2-1)(q-1)\delta_{1,x}$.

b) $c_2(x)$ equals $(1-q)\delta_{1,x}$.
c) $c_3(x,y)$ equals 0.
d) $c_4(z)$ equals 0.

5. Given a multiplicative character $\theta\colon \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ such that $\theta_{\mathbb{F}^1_{q^2}} \not\equiv 1$ (here $\mathbb{F}^1_{q^2} = \{z \in \mathbb{F}_{q^2} : z\bar{z} = 1\}$) we denote by π_θ the corresponding $(q-1)$-dimensional discrete series representation of G. There are $\frac{q^2-q}{2}$ pairwise non-isomorphic representations of this type. Prove that the character value of π_θ at
a) $c_1(x)$ equals $(q-1)\theta(x)$.
b) $c_2(x)$ equals $-\theta(x)$.
c) $c_3(x,y)$ equals 0.
d) $c_4(z)$ equals $-\theta(z)-\theta(\bar{z})$.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 07.03.2023

1. Recall the conjugacy classes in $\mathrm{SL}(2,\mathbb{F}_q)$ (Problem 3 of 17.01.2023). We denote them by $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_2 = -e_1$, e_3 (the class of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$), e_4 (the class of $\begin{pmatrix} 1 & \varepsilon \\ 1 & 0 \end{pmatrix}$, where $\varepsilon \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$), $e_5 = -e_3$, $e_6 = -e_4$, $c_3(x,x^{-1})$ (see Problem 2 of 28.02.2023), and $c_4(z)$ for $z \in \mathbb{F}^1_{q^2}$ (of norm 1). All the irreducible representations of $\mathrm{SL}(2,\mathbb{F}_q)$ are restricted from $\mathrm{GL}(2,\mathbb{F}_q)$ except for the four $\frac{q\pm1}{2}$-dimensional ones. Namely, $\mathrm{Ind}_B^G(\xi,1)^\pm$ (here ξ is the nontrivial character $\mathbb{F}_q^\times \to \{\pm 1\}$, and $\mathrm{Ind}_B^G(\xi,1)^\pm$ are eigenspaces of the Radon Transform with eigenvalues $\pm\sqrt{\xi(-1)}$), and $\pi_\tau^\pm$ (here τ is the nontrivial character $\mathbb{F}^1_{q^2} \to \{\pm1\}$, and $\pi_\tau|_{\mathrm{SL}(2,\mathbb{F}_q)} = \pi_\tau^+ \oplus \pi_\tau^-$). Our goal is to compute the characters $\chi_\xi^\pm$ and $\chi_\tau^\pm$ of these representations. First, prove that the values of these characters on the classes c_3 and c_4 are exactly one half of the values of the characters of $\mathrm{Ind}_B^G(\xi,1)$ and π_τ on these classes.

2. Prove that a) the values of $\chi_\xi^\pm$ (resp. $\chi_\tau^\pm$) at e_3, e_5 and also at e_4, e_6, are related by multiplication by $\xi(-1)$ (resp. $\tau(-1)$).
b) $\chi_\xi^\pm(e_3) = \chi_\xi^\mp(e_4)$, and $\chi_\tau^\pm(e_3) = \chi_\tau^\mp(e_4)$.
c) $\chi_\xi^\pm(e_3) = \frac{1\pm\sqrt{\xi(-1)q}}{2}$, and $\chi_\xi^\pm(e_4) = \frac{1\mp\sqrt{\xi(-1)q}}{2}$, while $\chi_\tau^\pm(e_3) = \frac{-1\pm\sqrt{\xi(-1)q}}{2} = \frac{-1\pm\sqrt{-\tau(-1)q}}{2}$, and $\chi_\tau^\pm(e_4) = \frac{-1\mp\sqrt{\xi(-1)q}}{2} = \frac{-1\mp\sqrt{-\tau(-1)q}}{2}$.

3. We consider the 2-dimensional vector space $U = \mathbb{F}_q^2$ equipped with the hyperbolic symmetric bilinear form $B((x_1,y_1),(x_2,y_2)) = x_1y_2 + x_2y_1$. The multiplicative group $\mathbb{F}_q^\times$ acts on $\mathbb{F}_q^2$ by $c(x,y) = (cx, c^{-1}y)$ and preserves B. We also consider $V = \mathbb{F}_q^2$ equipped with a symplectic form $\langle,\rangle$, and $W = V \otimes U$ equipped with the tensor product symplectic form. Let $\Lambda \subset W$ be a Lagrangian subspace of the form $\ell \otimes U$. Then we have the Weil representation of $\mathrm{Sp}(W) \circlearrowright \mathbb{C}[\Lambda]$, and we restrict it to $\mathrm{Sp}(V) \times \mathbb{F}_q^\times \subset \mathrm{Sp}(W)$. Given any character μ of $\mathbb{F}_q^\times$, the group $\mathrm{Sp}(V) \simeq \mathrm{SL}(2,\mathbb{F}_q)$ acts in the eigenspace $\mathbb{C}[\Lambda]_\mu$. Prove that if $\mu^2 \not\equiv 1$, then $\mathbb{C}[\Lambda]_\mu$

is an irreducible $(q+1)$-dimensional representation of $\mathrm{SL}(2,\mathbb{F}_q)$ induced from the character μ of its Borel subgroup.

4. Decompose $\mathbb{C}[\Lambda]_\mu$ into irreducibles for $\mu\colon \mathbb{F}_q^\times \to \{\pm 1\}$.

5. Extend $\mathbb{C}[\Lambda]_\mu$ to a representation of $\mathrm{GL}(2,\mathbb{F}_q)$ (choosing an extension of μ to a character $(\mu_1,\mu_2)\colon (\mathbb{F}_q^\times)^2 \to \mathbb{C}^\times$).

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 14.03.2023

1. In notation of Problem 1 of 07.03.2023, consider the representation π_θ of $\mathrm{SL}(2,\mathbb{F}_3)$ (where θ is a character $\mathbb{F}_9^1 \to \mathbb{C}^\times,\ \theta^2 \not\equiv 1$). Prove that

a) the image of π_θ is contained in $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C})$.

b) The kernel of $\pi_\theta\colon \mathrm{SL}(2,\mathbb{F}_3) \to \mathrm{SL}(2,\mathbb{C})$ is trivial. Thus, π_θ is an embedding $\mathrm{SL}(2,\mathbb{F}_3) \hookrightarrow \mathrm{SL}(2,\mathbb{C})$.

Recall from Problem 1 of 17.01.2023 that the image of $\mathrm{SL}(2,\mathbb{F}_3)$ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to the alternating group $\mathfrak{A}_4$.

2. Let $g \in \mathrm{SL}(2,\mathbb{F}_3)$ be an elliptic element (note that all the elliptic elements form a single conjugacy class), and let $\zeta \in \mathrm{SL}(2,\mathbb{C})$ be a square root of $\pi_\theta(g)$. Prove that $\zeta\,\{\pi_\theta(\mathrm{SL}(2,\mathbb{F}_3))\}\,\zeta^{-1} = \pi_\theta(\mathrm{SL}(2,\mathbb{F}_3))$, so that the subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ generated by $\pi_\theta(\mathrm{SL}(2,\mathbb{F}_3))$ and ζ, has order 48.

3. Construct an isomorphism $\Gamma/\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\} \simeq \mathrm{PSL}(2,\mathbb{Z}/4\mathbb{Z})$. Recall from Problem 2 of 17.01.2023 that the image $\mathrm{PSL}(2,\mathbb{Z}/4\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z}/4\mathbb{Z})$ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to the symmetric group $\mathfrak{S}_4$.

4. In notation of Problem 1 of 07.03.2023, consider the representation π_τ^+ of $\mathrm{SL}(2,\mathbb{F}_5)$ (where τ is a nontrivial character $\mathbb{F}_{25}^1 \to \mathbb{C}^\times,\ \tau^2 = 1$). Prove that

a) the image of π_τ^+ is contained in $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C})$.

b) The kernel of $\pi_\tau^+\colon \mathrm{SL}(2,\mathbb{F}_5) \to \mathrm{SL}(2,\mathbb{C})$ is trivial. Thus, π_τ^+ is an embedding $\mathrm{SL}(2,\mathbb{F}_5) \hookrightarrow \mathrm{SL}(2,\mathbb{C})$.

Recall from Problem 1 of 17.01.2023 that the image of $\mathrm{SL}(2,\mathbb{F}_5)$ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to the alternating group $\mathfrak{A}_5$.

5. For $n \geq 4$, consider the binary dihedral subgroup $\mathbb{D}_{n-2} \subset \mathrm{SL}(2,\mathbb{C})$ of order $4(n-2)$, generated by $\begin{pmatrix}\exp(\frac{2\pi\sqrt{-1}}{2n-4}) & 0\\ 0 & \exp(-\frac{2\pi\sqrt{-1}}{2n-4})\end{pmatrix}$ and by $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. The tautological embedding $\mathbb{D}_{n-2} \hookrightarrow \mathrm{SL}(2,\mathbb{C})$ is denoted π_1. Prove that apart from π_1, the group $\mathbb{D}_{n-2}$ has $n-4$ irreducible 2-dimensional representations $\pi_2,\dots,\pi_{n-3}$, and four 1-dimensional characters $\mathbb{C},\chi_1,\chi_2,\chi_3$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 21.03.2023

1. We know from Problem 1 of 07.03.2023 that $\mathrm{SL}(2, \mathbb{F}_3)$ has 7 irreducible representations $\mathbb{C}, \mathrm{St}, \mathrm{Ind}_B^G(\xi, 1)^{\pm}, \pi_\tau^{\pm}, \pi_\theta$ of dimensions $1, 3, 2, 2, 1, 1, 2$. We form the *McKay graph* with vertices labeled by these representations and edges $V_1 \longrightarrow V_2$ if V_2 is a summand of $V_1 \otimes \pi_\theta$. Prove that

a) $V_1 \longrightarrow V_2$ iff $V_2 \longrightarrow V_1$. Thus we obtain not a quiver (with directed arrows), but a graph (with non-directed edges denoted $V_1 = V_2$ in place of $V_1 - V_2$ because I am too lazy to draw tikz diagrams).

b) We get the graph

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[d, equal] & & \\
 & & \pi_\theta \arrow[d, equal] & & \\
\pi_\tau^+ \arrow[r, equal] & \mathrm{Ind}_B^G(\xi, 1)^+ \arrow[r, equal] & \mathrm{St} \arrow[r, equal] & \mathrm{Ind}_B^G(\xi, 1)^- \arrow[r, equal] & \pi_\tau^-
\end{tikzcd}$$

2. Recall the subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ of Problem 2 of 14.03.2023. We will denote its normal subgroup $\mathrm{SL}(2, \mathbb{F}_3)$ of index 2 by H. Prove that Γ has the following 8 irreducible representations:

$\mathrm{Ind}_H^\Gamma(\mathbb{C}) = \mathbb{C} \oplus \chi$ for a 1-dimensional character χ.

Furthermore, $\mathrm{Ind}_H^\Gamma(\pi_\tau^+) = \mathrm{Ind}_H^\Gamma(\pi_\tau^-) =: \rho \simeq \rho \otimes \chi$ is a 2-dimensional representation.

Furthermore, $\mathrm{Ind}_H^\Gamma(\pi_\theta) = \pi \oplus \pi \otimes \chi$, where π is the tautological 2-dimensional representation of Γ.

Furthermore, $\mathrm{Ind}_H^\Gamma(\mathrm{St}) = \sigma \oplus \sigma \otimes \chi$ for a 3-dimensional representation σ.

Finally, $\mathrm{Ind}_H^\Gamma(\mathrm{Ind}_B^G(\xi, 1)^+) = \mathrm{Ind}_H^\Gamma(\mathrm{Ind}_B^G(\xi, 1)^-) =: \varrho \simeq \varrho \otimes \chi$ for a 4-dimensional representation ϱ.

3. Prove that the McKay graph of Γ (with respect to tensoring with the tautological representation π) is

$$\begin{tikzcd}
 & & & \rho \arrow[d, equal] & & & \\
\chi \arrow[r, equal] & \pi \otimes \chi \arrow[r, equal] & \sigma \otimes \chi \arrow[r, equal] & \varrho \arrow[r, equal] & \sigma \arrow[r, equal] & \pi \arrow[r, equal] & \mathbb{C}
\end{tikzcd}$$

4. We know from Problem 1 of 07.03.2023 that $\mathrm{SL}(2, \mathbb{F}_5)$ has 9 irreducible representations $\mathbb{C}, \mathrm{St}, \mathrm{Ind}_B^G(\mu, 1), \mathrm{Ind}_B^G(\xi, 1)^{\pm}, \pi_\eta, \pi_\theta, \pi_\tau^{\pm}$ of dimensions $1, 5, 6, 3, 3, 4, 4, 2, 2$. Prove that the McKay graph of $\mathrm{SL}(2, \mathbb{F}_5)$ (with respect to tensoring with the tautological representation π_τ^+) is

$$\begin{tikzcd}[column sep=small]
 & & \operatorname{Ind}_B^G(\xi,1)^- \arrow[d, equal] & & & & & \\
\pi_\tau^- \arrow[r, equal] & \pi_\eta \arrow[r, equal] & \operatorname{Ind}_B^G(\mu,1) \arrow[r, equal] & \mathrm{St} \arrow[r, equal] & \pi_\theta \arrow[r, equal] & \operatorname{Ind}_B^G(\xi,1)^+ \arrow[r, equal] & \pi_\tau^+ \arrow[r, equal] & \mathbb{C}
\end{tikzcd}$$

5. Prove that the McKay graph of $\mathbb{D}_{n-2}$ (with respect to tensoring with π_1, notation of Problem 5 of 14.03.2023) is

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, equal] & & \chi_3 \arrow[d, equal] & \\
\chi_2 \arrow[r, equal] & \pi_1 \arrow[r, equal] & \cdots \arrow[r, equal] & \pi_{n-3} \arrow[r, equal] & \chi_1
\end{tikzcd}$$

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 28.03.2023

1. Prove a) $\displaystyle\prod_{i,j}(1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda^t}(y)$.

b) $\displaystyle E(t)^n = \sum_\lambda s_\lambda(x) s_{\lambda^t}(y) = \sum_\lambda \binom{n}{\lambda} s_\lambda(x) t^{|\lambda|}$ (where $E(t) = \sum e_r t^r = \prod(1 + x_i t)$, we set $y_1 = \ldots = y_n = t$, and $0 = y_{n+1} = y_{n+2} = \ldots$ Furthermore, $\displaystyle\binom{a}{\lambda} := \prod_{x\in\lambda} \frac{a - c(x)}{h(x)}$, and $c(x)$ is the content of x, and $h(x)$ is the hook length of x).

c) $\displaystyle H(t)^n = \sum_\lambda \binom{n}{\lambda^t} s_\lambda t^{|\lambda|}$ (where $H(t) = \sum h_r t^r = \prod(1 - x_i t)^{-1}$).

2. We set $y_i = q^{i-1},\ 1 \le i \le n$, and $y_i = 0,\ i > n$. Prove

a) $\displaystyle\prod_{i=1}^n E(q^{i-1}) = \sum_\lambda q^{n(\lambda^t)} \begin{bmatrix} n \\ \lambda \end{bmatrix} s_\lambda$, where $\displaystyle\begin{bmatrix} n \\ \lambda \end{bmatrix} := \prod_{x\in\lambda} \frac{1 - q^{n-c(x)}}{1 - q^{h(x)}}$.

b) $\displaystyle\prod_{i=1}^n H(q^{i-1}) = \sum_\lambda q^{n(\lambda)} \begin{bmatrix} n \\ \lambda^t \end{bmatrix} s_\lambda$.

c) $\displaystyle\prod_{i,j\ge 1}(1 + x_j q^{i-1}) = \sum_\lambda \frac{q^{n(\lambda^t)}}{H_\lambda(q)} s_\lambda(x)$.

d) $\displaystyle\prod_{i,j\ge 1}(1 - x_j q^{i-1})^{-1} = \sum_\lambda \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(x)$,

where $\displaystyle H_\lambda(q) = \prod_{x\in\lambda}(1 - q^{h(x)})$ is the hook polynomial.

3. We set $y_1 = \ldots = y_n = t/n,\ y_i = 0,\ i > n$, and take the limit as $n \to \infty$. Prove

a) $\frac{1}{n^{|\lambda|}}\binom{n}{\lambda} \to \prod_{x\in\lambda} h(x)^{-1} =: h(\lambda)^{-1}$.

b) $\prod_i \left(1+\frac{x_i t}{n}\right)^n \to \prod_i \exp(x_i t) = \exp(e_1 t) = \sum_\lambda \frac{s_\lambda}{h(\lambda)} t^{|\lambda|}$.

c) $e_1^n = \sum_{|\lambda|=n} \frac{n!}{h(\lambda)} s_\lambda \Leftrightarrow \langle e_1^n, s_\lambda\rangle = n!/h(\lambda)$.

4. Prove that the number of standard tableaux of shape $\lambda \in \mathcal{P}(n)$ equals $K_{\lambda,(1^n)} = \langle s_\lambda, h_1^n\rangle = n!/h(\lambda)$.

5. Prove that $\langle h_n, p_\lambda\rangle = 1$ and $\langle e_n, p_\lambda\rangle = \varepsilon_\lambda := (-1)^{|\lambda|+\ell(\lambda)}$ for any $\lambda \in \mathcal{P}(n)$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 04.04.2023

1. We identify $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with the ring of symmetric functions in variables x, y: $f \otimes g \mapsto f(x)g(y)$. We define a coproduct $\Delta\colon \Lambda \to \Lambda \otimes_{\mathbb{Z}} \Lambda$ by $(\Delta f)(x,y) = f(x,y)$. We define a counit $\varepsilon\colon \Lambda \to \mathbb{Z}$ requiring that $\varepsilon(\Lambda^n) = 0$ for $n > 0$, and $\varepsilon(1) = 1$. Prove that

a) $\Delta h_n = \sum_{0\le k\le n} h_k \otimes h_{n-k}$.

b) $\Delta e_n = \sum_{0\le k\le n} e_k \otimes e_{n-k}$.

c) $\Delta p_n = p_n \otimes 1 + 1 \otimes p_n$ (i.e. p_n are primitive).

d) $\Delta s_\lambda = \sum_\mu s_{\lambda/\mu} \otimes s_\mu$.

2. We equip $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with a scalar product such that $\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle = \langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle$. Prove that $\Delta\colon \Lambda \to \Lambda \otimes_{\mathbb{Z}} \Lambda$ is adjoint to the multiplication $\mathsf{m}\colon \Lambda \otimes_{\mathbb{Z}} \Lambda \to \Lambda$, and the counit $\varepsilon\colon \Lambda \to \mathbb{Z}$ is adjoint to the unit $e\colon \mathbb{Z} \to \Lambda$. In other words, the Hopf algebra Λ is selfdual.

3. For any $f \in \Lambda$ we define $D(f)\colon \Lambda \to \Lambda$ by $\langle D(f)u, v\rangle = \langle u, fv\rangle$ for any $u, v \in \Lambda$. Then $D\colon \Lambda \to \mathrm{End}(\Lambda)$ is a ring homomorphism. We denote $D(s_\mu)$ by D_μ. Prove that

a) for any $f \in \Lambda,\ f(x,y) = \sum_\mu D_\mu f(x) \cdot s_\mu(y)$.

b) $D(h_\lambda)m_\mu = 0$ unless $\mu = \lambda \cup \nu$ (that is, μ is the union of reordered parts of λ and ν), in which case $D(h_\lambda)m_\mu = m_\nu$.

c) For any $f(x_0, x_1, \ldots) \in \Lambda,\ (D(h_n)f)(x_1, x_2, \ldots)$ is the coefficient of x_0^n in f.

d) $D(f)(gh) = \sum_i (D(f_i^{(1)})g) \cdot (D(f_i^{(2)})h)$, where $\Delta f = \sum_i f_i^{(1)} \otimes f_i^{(2)}$.

4. Prove that a) $D(p_n)h_N = h_{N-n}$, that is $D(p_n) = \sum_{r\ge 0} h_r \frac{\partial}{\partial h_{n+r}}$, where we view the symmetric functions as polynomials in $h_i,\ i \ge 0$.

b) $D(p_n) = (-1)^{n-1} \sum_{r\ge 0} e_r \frac{\partial}{\partial e_{n+r}}$.

c) $D(p_n) = n\frac{\partial}{\partial p_n}$. In other words, if $f \in \Lambda$ is written as a polynomial $f = \varphi(p_1, p_2, \ldots)$, then $D(f) = \varphi(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots)$ is a linear differential operator with constant coefficients.

5. Define an involution $\tilde\omega = (-1)^n \omega$ on Λ^n. Prove that

a) $\tilde\omega$ is an antipode, i.e. $\mathsf{m} \circ (\tilde\omega \otimes \mathrm{Id}) \circ \Delta = \mathsf{m} \circ (\mathrm{Id} \otimes \tilde\omega) \circ \Delta = e \circ \varepsilon\colon \Lambda \to \Lambda$.

b) Any primitive element $p \in \Lambda^n$ (i.e. $\Delta p = p \otimes 1 + 1 \otimes p$) is proportional to p_n.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 11.04.2023

1. Prove that the character of the parabolic induction $\mathrm{Ind}_{P_{k,\ell}}^{\mathrm{GL}(k+\ell,\mathbb{F}_q)}(V_k\otimes V_\ell)$ is equal to the Hall product of characters $\chi(V_k)\in\mathbb{C}[\mathrm{GL}(k,\mathbb{F}_q)]^{\mathrm{GL}(k,\mathbb{F}_q)}$ and $\chi(V_\ell)\in\mathbb{C}[\mathrm{GL}(\ell,\mathbb{F}_q)]^{\mathrm{GL}(\ell,\mathbb{F}_q)}$ (viewed as functions on the set of isomorphism classes of representations of $\mathbb{F}_q[t^{\pm1}]$).

2. Prove that for a partition $\lambda=(i^{m_i})=(\lambda_1\geq\lambda_2\geq\ldots)$ we have

$$q^{|\lambda|+2n(\lambda)}\prod\varphi_{m_i}(\lambda)(q^{-1})=\prod q^{\lambda_1^t+\ldots+\lambda_r^t}(1-q^{\nu_r^t-\lambda_r^t}),$$

where the second product is taken over $r=\lambda_1,\lambda_2,\ldots$, and $\nu=(\lambda_1,\ldots,\lambda_{k-1})$ for $r=\lambda_k$. Furthermore, λ^t stands for the dual partition (corresponding to the transposed Young diagram), and $\varphi_m(t):=(1-t)\cdots(1-t^m)$.

3. Construct a bijection between the set of partitions λ whose Young diagram is contained in the $k\times\ell$-box and the set of sequences of nonnegative integers $(a_1,\ldots,a_m;b_0,\ldots,b_m)$ such that $\sum a_i=k$, $\sum b_j=\ell$, and $a_1,\ldots,a_m,b_1,\ldots,b_{m-1}$ are all positive, but b_0 and b_m can possibly vanish.

4. Fix a complete flag $0=V_0\subset V_1\subset\ldots\subset V_{k+\ell}=\mathbb{C}^{k+\ell}$. We define the *Schubert cell* $X_\lambda\subset\mathrm{Gr}(k,k+\ell)$ as the set of all k-dimensional subspaces $U\subset\mathbb{C}^{k+\ell}$ such that

$$\dim(U\cap V_{b_0})=0,\ \dim(U\cap V_{b_0+a_1})=a_1,$$

$$\vdots$$

$$\dim(U\cap V_{b_0+a_1+\ldots+b_{i-2}+a_{i-1}+b_{i-1}})=a_1+\ldots+a_{i-1},\ \dim(U\cap V_{b_0+a_1+\ldots+b_{i-1}+a_i})=a_1+\ldots+a_i,$$

$$\vdots$$

$$\dim(U\cap V_{b_0+a_1+\ldots+b_{m-2}+a_{m-1}+b_{m-1}})=a_1+\ldots+a_{m-1},\ \dim(U\cap V_{b_0+a_1+\ldots+b_{m-1}+a_m})=a_1+\ldots+a_m.$$

Prove that a) $\mathrm{Gr}(k,k+\ell)=\bigsqcup_\lambda X_\lambda$.

b) X_λ is an orbit in $\mathrm{Gr}(k,k+\ell)$ of the Borel subgroup of $\mathrm{GL}(k+\ell,\mathbb{C})$ preserving the above complete flag.

c) $X_\mu\subset\overline{X}_\lambda$ iff $\mu\subset\lambda$, i.e. the Young diagram of μ is contained in the Young diagram of λ.

5. Construct an isomorphism $X_\lambda\simeq\mathbb{C}^{|\lambda|}$.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 18.04.2023

1. (a) For partitions λ,μ, we denote by $\lambda\mu$ (resp. $\lambda\otimes\mu$) a partition with parts $\lambda_i\mu_i$ (resp. $\min(\lambda_i,\mu_j)$). Prove that $(\lambda\mu)^t=\lambda^t\otimes\mu^t$.

(b) Let M,N be $\mathcal{O}$-modules of types μ,ν. Prove that the type of $M\oplus N$ (resp. $M\otimes N$) is $\mu\cup\nu$ (resp. $\mu\otimes\nu$).

2. Prove that the structure constant in the Hall algebra $G^\lambda_{\mu(1^m)}(q)=q^{n(\lambda)-n(\mu)-n(1^m)}\prod_{i\geq1}\begin{bmatrix}\lambda_i^t-\lambda_{i+1}^t\\ \lambda_i^t-\mu_i^t\end{bmatrix}_{q^{-1}}$.

3. Prove (a) $R_\lambda(1,t,\dots,t^{n-1};t)=t^{n(\lambda)}v_n(t)$, where $v_n(t)=\prod_{i=1}^n\frac{1-t^i}{1-t}$.

(b) $Q_\lambda(1,t,\dots,t^{n-1};t)=t^{n(\lambda)}\varphi_n(t)/\varphi_{m_0}(t)$, where $m_0=n-\ell(\lambda)$, and $\varphi_n(t)=v_n(t)(1-t)^n$. As $n\to\infty$, we get in the limit $Q_\lambda(1,t,t^2,\dots;t)=t^{n(\lambda)}$.

4. Prove

(a) $P_\lambda(x_1,\dots,x_n;t)=v_\lambda(t)^{-1}\prod_{i<j}(1-tR_{ji})s_\lambda(x_1,\dots,x_n)=\prod_{\lambda_i>\lambda_j}(1-tR_{ji})s_\lambda(x_1,\dots,x_n)$, where $v_\lambda(t)=\prod_{i\ge 0}v_{m_i}(t)$ for $\lambda=(i^{m_i})$ (starting from $i=0$, so that $m_0=n-\ell(\lambda)$), and R_{ji} are the raising operators.

(b) $P_{(n)}=\sum_{r=0}^{n-1}(-t)^r s_{(n-r,1^r)}$.

5. Prove (a) $\displaystyle\sum_{i=1}^n\prod_{j\neq i}\frac{x_j-tx_i}{x_j-x_i}=\frac{v_n(t)}{v_{n-1}(t)}=\frac{1-t^n}{1-t}$.

(b) $\displaystyle\sum_{i=1}^n\prod_{j\neq i}\left(1-\frac{x_i}{x_j}\right)^{-1}=1$.

(c) Let $(a_1,\dots,a_n)\in\mathbb{N}^n$. We define $c(a_1,\dots,a_n)$ as the constant term of $\displaystyle\prod_{1\le i\neq j\le n}\left(1-\frac{x_j}{x_i}\right)^{a_j}$.

Then $c(a_1,\dots,a_n)=\displaystyle\sum_{i=1}^n c(a_1,\dots,a_i-1,\dots,a_n)$.

(d) $c(a_1,\dots,a_n)=(a_1+\ldots+a_n)!/a_1!\cdots a_n!$.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 25.04.2023

1. For the structure constants $P_\mu(x;t)P_\nu(x;t)=\sum_\lambda f^\lambda_{\mu\nu}(t)P_\lambda(x;t)$ prove

a) $\displaystyle\sum_\mu t^{n(\mu)}f^\lambda_{\mu(1^m)}(t)=t^{n(\lambda)-m(m-1)/2}\begin{bmatrix}\ell(\lambda)\\ m\end{bmatrix}(t^{-1})$.

b) $\displaystyle\left(\sum_\mu t^{n(\mu)}P_\mu\right)\left(\sum_m e_m y^m\right)=\sum_\lambda t^{n(\lambda)}P_\lambda\prod_{j=1}^{\ell(\lambda)}(1+t^{1-j}y)$.

c) $\displaystyle\left(\sum_\mu t^{n(\mu)}P_\mu\right)\left(\sum_m(-1)^m e_m\right)=1$.

d) $\displaystyle\sum_{|\mu|=r}t^{n(\mu)}P_\mu=h_r$.

2. Prove that the Kostka polynomial $K_{(r)\mu}(t)=t^{n(\mu)}$ for any $|\mu|=r$.

3. Set $S_\lambda(x;t):=\det\big(q_{\lambda_i-i+j}(x;t)\big)$. Prove that

a) $S_\lambda(x;t)=\prod_{i<j}(1-R_{ij})q_\lambda=\prod_{i<j}(1-tR_{ij})Q_\lambda$.

b) $\displaystyle\prod_{i,j}\frac{1-tx_iy_j}{1-x_iy_j}=\sum_\lambda S_\lambda(x;t)s_\lambda(y)=\sum_\lambda s_\lambda(x)S_\lambda(y;t)$.

c) $\langle S_\lambda(x;t), s_\mu(x)\rangle = \delta_{\lambda\mu}$.

4. Prove that a) $\displaystyle\sum_\lambda S_\lambda(x;t)s_{\lambda^t}(y) = \prod_{i,j}\frac{1+x_iy_j}{1+tx_iy_j}$.

b) If we specialize $y_i = t^{i-1}$, we get $s_{\lambda^t}(y) = t^{n(\lambda^t)}H_\lambda(t)^{-1}$ (the hook polynomial of Problem 2 of 28.03.2023).

c) $\displaystyle\sum_\lambda S_\lambda(x;t)t^{n(\lambda^t)}H_\lambda(t)^{-1} = \prod(1+x_i)$.

5. Prove that $K_{\lambda(1^r)}(t) = t^{n(\lambda^t)}\varphi_r(t)H_\lambda(t)^{-1}$ for any $|\lambda| = r$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 02.05.2023

1. Set $x_i = q^{i-1}$ for $1 \le i \le n$, and $x_i = 0$ for $i > n$. Prove that

a) $\displaystyle E(t) = \prod_{i=0}^{n-1}(1+q^it) = \sum_{r=0}^{n} q^{r(r-1)/2}\begin{bmatrix} n \\ r\end{bmatrix} t^r$.

b) $\displaystyle H(t) = \prod_{i=0}^{n-1}(1-q^it)^{-1} = \sum_{r=0}^{\infty}\begin{bmatrix} n+r-1 \\ r\end{bmatrix} t^r$.

2. Prove that $\displaystyle\prod_{i\ge1}\frac{1+x_iy}{1-x_i} = \sum_\lambda t^{n(\lambda)}\prod_{j=1}^{\ell(\lambda)}(1+t^{1-j}y)P_\lambda(x;t)$.

3. Prove a) $\displaystyle p_r(x) = \sum_{|\lambda|=r} t^{n(\lambda)}\prod_{i=1}^{\ell(\lambda)-1}(1-t^{-i})P_\lambda(x;t)$.

b) $X_{(n)\lambda}(t) = t^{n(\lambda)}\varphi_{\ell(\lambda)-1}(t^{-1})$; equivalently, $Q_{(n)\lambda}(q) = \varphi_{\ell(\lambda)-1}(q)$.

c) $X_{\rho(n)}(t) = 1 = Q_{\rho(n)}(q)$ for any partition $|\rho| = n$.

4. Prove that a) $\displaystyle\sum_\rho z_\rho(t)^{-1}X_{\rho(1^n)}(t)p_\rho(x) = \varphi_n(t)e_n(x) = \varphi_n(t)\sum_\rho \varepsilon_\rho z_\rho^{-1}p_\rho(x)$, where $\varepsilon_\rho = (-1)^{|\rho|-\ell(\rho)}$.

b) $\displaystyle X_{\rho(1^n)}(t) = \varepsilon_\rho z_\rho^{-1}z_\rho(t)\varphi_n(t) = \prod_{i=1}^{n}(t^i-1)/\prod_{j\ge1}(t^{\rho_j}-1)$; equivalently, $\displaystyle Q_{\rho(1^n)}(q) = \varphi_n(q)/\prod_{j\ge1}(1-q^{\rho_j})$.

5. Prove that a) $\displaystyle t^{n(\lambda)} = \sum_\rho z_\rho^{-1}X_{\rho\lambda}(t)$; equivalently, $\displaystyle\sum_{|\rho|=n} z_\rho^{-1}Q_{\rho\lambda}(q) = 1$.

b) $X_{\rho\lambda}(1) = \langle p_\rho, h_\lambda\rangle$.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 09.05.2023

1. We fix an identification $\mathsf{k}_n = \mathbb{F}_{q^n} \simeq \mathbb{F}_q^n = \mathsf{k}^n$. Then for any $x \in M_n := \mathsf{k}_n^\times$, the invertible operator of multiplication by x can be viewed as an element of $G_n := \mathrm{GL}(n,\mathbb{F}_q)$, and we obtain an injective homomorphism $M_n \hookrightarrow G_n$. Its image is denoted T_n (a non-split maximal torus), and if we change an identification $\mathsf{k}_n \simeq \mathsf{k}^n$, the torus T_n is replaced by its conjugate. For any $x \in M_n$, the eigenvalues of the corresponding (semisimple) element of $T_n \subset G_n$ are $\mathrm{Fr}^i x,\ 0 \le i \le n-1$. If the minimal polynomial of x is f_x of degree d_x, then the conjugacy class of the corresponding element of G_n is $\underline{\mu}$, where $\underline{\mu}(f_x) = (1^{n/d_x})$, and $\underline{\mu}(f) = 0$ for $f \ne f_x$.

For a partition $|\nu| = n$ we have a maximal torus $T_\nu = T_{\nu_1} \times \cdots \times T_{\nu_\ell} \subset G_{\nu_1} \times \cdots \times G_{\nu_\ell} \subset G_n$. Let $W_\nu := \mathrm{Norm}_{G_n}(T_\nu)/T_\nu$. Prove that W_ν is isomorphic to the centralizer in $\mathfrak{S}_n$ of an element of the cycle type ν. In particular, $W_{(n)} \cong \mathrm{Gal}(\mathsf{k}_n : \mathsf{k}) \simeq \mathbb{Z}/n\mathbb{Z}$.

2. Prove that the number of conjugacy classes in G_n is equal to $\sum_{|\nu|=n} \sharp(W_\nu \backslash T_\nu)$ (the number of orbits of W_ν in T_ν).

3. Prove that for a central function u on G_n, its value at the conjugacy class $c_{\underline{\mu}}$ is equal to $\langle ch(u), \widetilde{Q}_{\underline{\mu}} \rangle$.

4. Let $\nu = (\nu_1, \ldots, \nu_\ell),\ |\nu| = n$, and let θ_ν be a character of the torus $T_\nu \simeq M_{\nu_1} \times \cdots \times M_{\nu_\ell}$, that is $\theta_\nu = (\xi_1, \ldots, \xi_\ell)$, where $\xi_i \in L_{\nu_i} = M_{\nu_i}^\vee$. Then $\prod_{i=1}^\ell \tilde{p}_{\nu_i}(\xi_i)$ is the characteristic of a certain character $R_{T_\nu}^{\theta_\nu}$ of G_n (Green's *principal character*, or *Deligne-Lusztig induction*). Prove that

a) $R_{T_\nu}^{\theta_\nu}$ depends only on the Fr-orbit of ξ_i, i.e. on the W_ν-orbit of $\theta_\nu \in T_\nu^\vee$.

b) Distinct characters $R_{T_\nu}^{\theta_\nu}$ for $\theta_\nu \in \bigsqcup_\nu (W_\nu \backslash T_\nu^\vee)$ are all mutually orthogonal in the space $\mathcal{A}_n$ of central functions on G_n.

c) The characters $R_{T_\nu}^{\theta_\nu}$ in b) above form an orthogonal basis of $\mathcal{A}_n$.

d) Write down all the characters $R_{T_\nu}^{\theta_\nu}$ explicitly for $n = 2$ (say, as linear combinations of irreducible characters of Problem set of 28.02.2023).

5. Prove that a) $R_{T_\nu}^{\theta_\nu}$ is irreducible iff θ_ν is a *regular* character of T_ν, i.e. its stabilizer in W_ν is trivial.

b) The value of $R_{T_\nu}^{\theta_\nu}$ at the unipotent conjugacy class of Jordan type λ is $(-1)^{n-\ell} Q_{\nu\lambda}(q)$ (the Green function).

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 16.05.2023

1. Prove that the central character of the irreducible representation of G_n corresponding to $\underline{\lambda}\colon \Theta \to \mathcal{P}$, is $\Delta(\underline{\lambda})$ defined as follows. First, we view $\underline{\lambda}$ as a function on L: we set $\underline{\lambda}(\xi) := \underline{\lambda}(\phi)$, where ϕ is the Fr-orbit of $\xi \in L$. Then we set $\Delta(\underline{\lambda}) := \prod_{\phi \in \Theta} \xi_\phi^{|\lambda(\phi)|} \in L_n = M_n^\vee = (\mathsf{k}_n^\times)^\vee$, where ξ_ϕ is any representative of an orbit ϕ.

So a scalar matrix $a \cdot \mathrm{Id}_n \in G_n$ acts as multiplication by $\langle \Delta(\underline{\lambda}), a \rangle_n \in \mathbb{C}^\times$.

2. Let $U_n \subset G_n$ be the subset of unipotent elements. For $\underline{\lambda}\colon \Theta \to \mathcal{P},\ \|\underline{\lambda}\| = n$, we consider the corresponding irreducible character $\chi^{\underline{\lambda}}$ of G_n and set $E_{\underline{\lambda}} := \sum_{u\in U_n} \chi^{\underline{\lambda}}(u)$. Prove that

a) $E_{\underline{\lambda}} = \sharp(G_n) \sum_{|\rho|=n} a_\rho(q)^{-1}\langle S_{\underline{\lambda}}, \widetilde{Q}_\rho(f_1)\rangle = \sharp(G_n)\langle S_{\underline{\lambda}}, h_n(f_1)\rangle$, where $f_1 = t-1$.

b) $q^{n(n-1)/2}\psi_n(q)h_n(f_1) = \sum_{\|\underline{\lambda}\|=n} E_{\underline{\lambda}}\overline{S}_{\underline{\lambda}}$, where $\psi_n(q) = \prod_{i=1}^n (q^i-1) = (-1)^n\varphi_n(q)$.

c) $h_n(f_1) = \sum_{\|\underline{\lambda}\|=n} \varepsilon(S_{\underline{\lambda}})\overline{S}_{\underline{\lambda}}$, where $\varepsilon\colon \mathcal{B} \to \mathbb{C}$ is the algebra homomorphism such that $\varepsilon(\tilde{p}_n(1)) = (q^n-1)^{-1}$, and $\varepsilon(\tilde{p}_n(x)) = 0$ for $x \neq 1$.

d) $E_{\underline{\lambda}} = (-1)^{a(\underline{\lambda})}q^{N(\underline{\lambda})}\chi^{\underline{\lambda}}(\mathrm{Id}_n)$, where $a(\underline{\lambda}) := n - \sum_\phi |\underline{\lambda}(\phi)|$, and $N(\underline{\lambda}) := n(n-1)/2 + \sum_\phi d(\phi)(n(\underline{\lambda}(\phi)) - n(\underline{\lambda}(\phi)^t))$.

3. Consider the algebra homomorphism $\delta\colon \mathcal{B} \to \mathbb{C}$ such that $\delta(\tilde{p}_n(1)) = (-1)^{n-1}(q^n-1)^{-1}$, and $\delta(\tilde{p}_n(x)) = 0$ for $x \neq 1$. We have $d_{\underline{\lambda}} = \chi^{\underline{\lambda}}(\mathrm{Id}_n) = \psi_n(q)\delta(S_{\underline{\lambda}})$. We set $S := \sum_{\underline{\lambda}} \delta(S_{\underline{\lambda}})t^{\|\underline{\lambda}\|}$. Prove that

a) $S = \prod_\phi \sum_\lambda \delta(s_\lambda(\phi))t^{|\lambda|\cdot d(\phi)}$.

b) $\log S = \sum_\phi \left(\sum_i \log(1-(tq^{-i})^{d(\phi)})^{-1} + \sum_{i<j}\log(1-(t^2q^{-i-j})^{d(\phi)})^{-1}\right)$

$$= \sum_{m\geq 1} d_m \left(\sum_{i\geq 1}\sum_{r\geq 1}\frac{(tq^{-i})^{mr}}{r} + \sum_{i<j}\sum_{r\geq 1}\frac{(t^2q^{-i-j})^{mr}}{r}\right) = \sum_{m\geq 1}\sum_{r\geq 1}\frac{d_m}{r}\frac{i^{mr}}{q^{mr}-1}\left(1+\sum_{i\geq 1}t^{mr}q^{-2imr}\right)$$

$$= \sum_{N\geq 1}\left(\frac{t^N}{N} + \sum_{i\geq 1}\frac{t^{2N}q^{-2iN}}{N}\right) = \log(1-t)^{-1} + \sum_{i\geq 1}\log(1-t^2q^{-2i})^{-1},$$

where d_m is the number of orbits ϕ of cardinality $d(\phi) = m$.

c) $S = \frac{1}{1-t}\prod_{i\geq 1}\frac{1}{1-t^2q^{2i}} = (1+t)\prod_{i\geq 0}\frac{1}{1-t^2q^{2i}} = (1+t)\sum_{m\geq 0}\frac{t^{2m}}{(1-q^{-2})\cdots(1-q^{-2m})}$.

4. Prove that the sum of dimensions of all irreducible representations of G_n is equal to

a) $\sum_{\|\underline{\lambda}\|=n} d_{\underline{\lambda}} = (q-1)q^2(q^3-1)q^4(q^5-1)\ldots$ (n factors altogether).

b) The number of symmetric matrices in G_n.

5. Prove that a) $\sum_{\|\underline{\mu}\|=2n} d_{\underline{\mu}} = (q-1)q^2(q^3-1)q^4(q^5-1)\ldots(q^{2n}-1)$, where the sum runs over all $\underline{\mu}$ such that $\underline{\mu}(\phi)^t$ is even for all ϕ.

b) This is equal to the number of nondegenerate skew-symmetric bilinear forms on k^{2n}. By the way, the sum $\sum_{\|\underline{\mu}\|=2n}\chi^{\underline{\mu}}$ (the summation runs over the same set as in a) above) is equal to the induced character $\mathrm{Ind}_{\mathrm{Sp}(2n,\mathbb{F}_q)}^{\mathrm{GL}(2n,\mathbb{F}_q)}(\mathbb{C})$. The fact that $\mathrm{Ind}_{\mathrm{Sp}(2n,\mathbb{F}_q)}^{\mathrm{GL}(2n,\mathbb{F}_q)}(\mathbb{C})$ has a simple spectrum follows from Problem 1b) of 07.02.2023.

Exercises on representations of $\mathrm{GL}(n,\mathbb{F}_q)$ 23.05.2023

1. Let $M_n \subset G_n = \mathrm{GL}(n,\mathsf{k})$ be the *mirabolic* subgroup formed by all the matrices with the last row $(0,\ldots,0,1)$. It is the group $G_{n-1} \ltimes \mathsf{k}^{n-1}$ of all the affine transformations of k^{n-1}. The group of characters $\mathsf{k}^{n-1} \to \mathbb{C}^\times$ has two orbits under the action of G_{n-1}: that of the trivial character 1, and that of a nontrivial character ψ. We have $\mathrm{Stab}_{G_{n-1}}\psi \simeq M_{n-1}$.

Consider the "induction" functors on representation categories $\mathrm{Ind}_{G_{n-1}}^{M_n}$ and $\mathrm{Ind}_{M_{n-1}}^{M_n,\psi}$ defined as follows. Given a representation of M_{n-1}, we extend it to a representation of $M_{n-1} \ltimes \mathsf{k}^{n-1}$ such that k^{n-1} acts via ψ, and define $\mathrm{Ind}_{M_{n-1}}^{M_n,\psi}$ as the induction of this extended representation from $M_{n-1} \ltimes \mathsf{k}^{n-1}$ to M_n. Similarly, given a representation of G_{n-1}, we extend it to a representation of $M_n = G_{n-1} \ltimes \mathsf{k}^{n-1}$ with the trivial action of k^{n-1}. We have the (two-sided) adjoint "restriction" functors $\mathrm{Res}_{M_n}^{G_{n-1}}$ and $\mathrm{Res}_{M_n,\psi}^{M_{n-1}}$: the former one sends a representation of M_n to its k^{n-1}-invariants acted upon by G_{n-1}, and the latter one sends a representation of M_n to its (k^n, ψ)-eigenspace acted upon by M_{n-1}.

Prove that $\mathrm{Res}_{M_n}^{G_{n-1}} \oplus \mathrm{Res}_{M_n,\psi}^{M_{n-1}} : \mathrm{Rep}(M_n) \to \mathrm{Rep}(G_{n-1}) \oplus \mathrm{Rep}(M_{n-1})$ and $\mathrm{Ind}_{G_{n-1}}^{M_n} \oplus \mathrm{Ind}_{M_{n-1}}^{M_n,\psi} : \mathrm{Rep}(G_{n-1}) \oplus \mathrm{Rep}(M_{n-1}) \to \mathrm{Rep}(M_n)$ are mutually inverse isomorphisms between the based Grothendieck groups of representation categories (with bases formed by the classes of irreducible representations).

2. We define $\mathrm{Res} = \mathrm{Res}_n : \mathrm{Rep}(M_n) \to \bigoplus_{k=1}^n \mathrm{Rep}(G_{n-k})$ and $\mathrm{Ind} = \mathrm{Ind}_n : \bigoplus_{k=1}^n \mathrm{Rep}(G_{n-k}) \to \mathrm{Rep}(M_n)$ by induction in n as follows: Res_n is the composition

$$\mathrm{Rep}(M_n) \xrightarrow{\mathrm{Res}_{M_n}^{G_{n-1}} \oplus \mathrm{Res}_{M_n,\psi}^{M_{n-1}}} \mathrm{Rep}(G_{n-1}) \oplus \mathrm{Rep}(M_{n-1}) \xrightarrow{\mathrm{Id} \oplus \mathrm{Res}_{n-1}} \mathrm{Rep}(G_{n-1}) \oplus \bigoplus_{k=1}^{n-1} \mathrm{Rep}(G_{n-1-k}),$$

and Ind_n is defined similarly.

Prove that Res and Ind are mutually inverse based isomorphisms between $\mathrm{Rep}(M_n)$ and $\bigoplus_{k=1}^n \mathrm{Rep}(G_{n-k})$. In particular, the irreducible representations of M_n are naturally numbered by $\underline{\lambda} : \Theta \to \mathcal{P}$ such that $\|\underline{\lambda}\| < n$.

3. Recall that $\bigoplus_n \mathrm{Rep}(G_n)$ forms a Hopf algebra $\mathcal{H}(\mathsf{k})$ with multiplication defined via parabolic induction, and comultiplication Δ defined via parabolic restriction. The Hopf algebra $\mathcal{H}(\mathsf{k})$ is the tensor product of many copies of Λ (see the Problem set of 04.04.2023) numbered by Φ. Let $\delta : \mathcal{H}(\mathsf{k}) \to \mathbb{Z}$ be the additive homomorphism that sends any representation of G_n to the dimension of its (U_n, ψ_n)-eigenspace. Here $U_n \subset G_n$ is the subgroup of strictly upper-triangular matrices, and $\psi_n : U_n \to \mathbb{C}^\times$ is the product of additive characters $\psi : \mathsf{k} \to \mathbb{C}^\times$ applied to the elements right above the diagonal of an upper-triangular matrix. We define $\mathcal{D} : \mathcal{H}(\mathsf{k}) \to \mathcal{H}(\mathsf{k})$ as the composition

$$\mathcal{H}(\mathsf{k}) \xrightarrow{\Delta} \mathcal{H}(\mathsf{k}) \otimes \mathcal{H}(\mathsf{k}) \xrightarrow{\mathrm{Id} \otimes \delta} \mathcal{H}(\mathsf{k}).$$

Prove that a) δ is an algebra homomorphism $\mathcal{H}(\mathsf{k}) \to \mathbb{Z}$.

b) The composition $\mathrm{Res} \circ \mathrm{Res}_{G_n}^{M_n} : \mathrm{Rep}(G_n) \to \bigoplus_{k=1}^n \mathrm{Rep}(G_{n-k})$ is equal to $\mathcal{D} - 1$.

c) The composition $\mathrm{Res} \circ \mathrm{Ind}_{G_{n-1}}^{M_n} : \mathrm{Rep}(G_{n-1}) \to \bigoplus_{k=1}^n \mathrm{Rep}(G_{n-k})$ is equal to $\mathcal{D}$.

4. a) Prove that for an irreducible character $\chi^{\underline{\lambda}}$ of G_n, its restriction to M_n is the direct sum of $\mathrm{Ind}_n(\chi^{\underline{\mu}})$ over all $\underline{\mu}$ such that for any $\phi \in \Theta$ the corresponding $\underline{\mu}(\phi)$ is a little bit smaller than $\underline{\lambda}(\phi)$ (notation: $\underline{\mu} \dashv \underline{\lambda} \Leftrightarrow \underline{\mu}(\phi) \dashv \underline{\lambda}(\phi)\ \forall \phi$), that is for any i, $\underline{\lambda}(\phi) - 1 \leq \underline{\mu}(\phi) \leq \underline{\lambda}(\phi)$. In particular, $\chi^{\underline{\lambda}}|_{M_n}$ has a simple spectrum.

b) Prove that the convolution algebra of complex functions on $M_n\backslash G_n/M_n$ is commutative.

5. a) Prove that for an irreducible character $\mathrm{Ind}_n(\chi^{\underline{\lambda}})$, $\|\underline{\lambda}\|< n$ of M_n, its restriction to G_{n-1} is the direct sum of $\chi^{\underline{\mu}}$ over all $\underline{\mu}$, $\|\underline{\mu}\|= n-1$ such that $\underline{\lambda} \dashv \underline{\mu}$. In particular, $\mathrm{Ind}_n(\chi^{\underline{\lambda}})|_{G_{n-1}}$ has a simple spectrum.

b) Prove that the convolution algebra of complex functions on $G_{n-1}\backslash M_n/G_{n-1}$ is commutative.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 30.05.2023

1. Given an ADE quiver Q, we say that a sequence of vertices $(i_\nu, \ldots, i_1)$ is $+$-admissible if i_ν is a source of Q, and $i_{\nu-1}$ is a source of the reflected quiver $\sigma_{i_\nu}Q$ (where we changed the orientations of arrows going from the vertex i_ν), and so on. Let $\nu = \sharp(R^+) = \ell(w_0)$. Prove that

a) there is a $+$-admissible sequence $(i_\nu, \ldots, i_1)$ such that $w_0 = s_{i_1} \cdots s_{i_\nu}$.

b) The set $\{\alpha^1, \ldots, \alpha^\nu\}$ (where $\alpha^k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k}$) coincides with R^+.

c) The set $\{V^1, \ldots, V^\nu\}$ (where $V^k = R^+_{i_1} \circ R^+_{i_2} \circ \cdots \circ R^+_{i_{k-1}} V_{i_k}$) contains all the indecomposable representations of Q.

d) For any $1 \le a < b \le \nu$, we have $\mathrm{Hom}(V^b, V^a) = 0 = \mathrm{Ext}^1(V^a, V^b)$.

2. Prove that any two products of all the simple reflections in a Weyl group W (in an arbitrary order) are conjugate. Such a product is called a *Coxeter element* $c \in W$. Its order is called the *Coxeter number* h, and its eigenvalues in the reflection representation $W \circlearrowright \mathfrak{h}^*$ are the roots of unity $\exp\left(\frac{2\pi\sqrt{-1}m_k}{h}\right)$, where $0 \le m_1 \le \ldots \le m_r < h$ are called the *exponents* of W (and r is the rank of $\mathfrak{h}$).

3. Let us color the vertices of Q black and white, so that the neighboring vertices have different colors. Denote by s' (resp. s'') the product of all the black (resp. white) simple reflections (note that all the reflections of the same color commute with each other), so that $c = s's''$ is a Coxeter element. Prove that

a) $0 < m_1,\ m_j + m_{r+1-j} = h$, and $m_1 + \ldots + m_r = rh/2$.

b) There are two linearly independent vectors $v', v'' \in \mathfrak{h}^*$ such that $s'(\mathbb{R}v' \oplus \mathbb{R}v'') = s''(\mathbb{R}v' \oplus \mathbb{R}v'') = \mathbb{R}v' \oplus \mathbb{R}v''$, and s', s'' restricted to this *Coxeter plane* are the orthogonal reflections with respect to the lines $\mathbb{R}v', \mathbb{R}v''$.

c) The Coxeter element $c = s's''$ restricted to the Coxeter plane $\mathbb{R}v' \oplus \mathbb{R}v''$ is the rotation with the angle $2\pi/h$.

d) The vectors v', v'' lie in the closure of the fundamental Weyl chamber, and the intersection of the Coxeter plane with the open fundamental Weyl chamber is $\mathbb{R}_+v' \oplus \mathbb{R}_+v''$.

4. Prove that a) $m_1 = 1,\ m_r = h-1$.

b) $2\nu = \sharp(R) = rh$.

c) For any $v \in \mathfrak{h}^*$ we have $\sum_{\alpha\in R}(v,\alpha)^2 = 2h(v,v)$ (we assume that for any root $\alpha \in R$ we have $(\alpha,\alpha) = 2$).

d) Let the highest root $\theta = \sum_{i=1}^{r} a_i\alpha_i$ (coefficients in the basis of simple roots). Then $\sum_{i=1}^{r} a_i = h - 1$.

5. a) Compute the Coxeter numbers for all the simple root systems (at least, ADE).

b) Prove that if h is even, then $w_0 = c^{h/2}$.

Exercises on representations of $\mathrm{GL}(n, \mathbb{F}_q)$ 06.06.2023

1. Let D be a central simple algebra (e.g. $D = \mathbb{H}_{a,b}$ a quaternion algebra with a basis $1, i, j, k$ and relations $i^2 = a,\ j^2 = b,\ k = ij = -ji$) over a local field $\mathcal{K}$ (e.g. $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbb{Q}_p$). Let $N\colon D \to \mathcal{K}$ be the norm map (a multiplicative homomorphism) (e.g. $N(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2$). Prove that D is a skew-field (a division algebra) iff $N^{-1}(1)$ is compact.

2. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of smooth functions all of whose derivatives are rapidly decreasing at infinity (e.g. $f(x) = \exp(-x^2)$) with its natural topology given by a system of norms $\| f \|_{a,b} = \sup_{x\in\mathbb{R}} |x^a \partial_x^b f|$. Let $\mathcal{S}(\mathbb{R})^\vee$ be the dual topological space of distributions of tempered growth (e.g. $\Delta = \sum_{n\in\mathbb{Z}} \delta_n \in \mathcal{S}(\mathbb{R})^\vee$). Prove that Δ is a unique distribution in $\mathcal{S}(\mathbb{R})^\vee$ (up to a multiple) invariant under multiplication by $\exp(2\pi\sqrt{-1}x)$ and under translations $f(x) \mapsto f(x+1)$.

3. Prove that the Heisenberg group $\mathrm{Heis}_{\mathbb{Q}} \subset \mathrm{Heis}_{\mathbb{A}}$ acting on the Schwartz space $\mathcal{S}(\mathbb{A})$ and its dual $\mathcal{S}(\mathbb{A})^\vee$ has a unique (up to a multiple) invariant vector $w \in \mathcal{S}(\mathbb{A})^\vee$. Here $\mathcal{S}(\mathbb{A})$ is the restricted tensor product $\mathcal{S}(\mathbb{R}) \otimes \bigotimes_p' \mathcal{S}(\mathbb{Q}_p)$ over all places of $\mathbb{Q}$ with respect to the system of vectors $\delta_{\mathbb{Z}_p} \in \mathcal{S}(\mathbb{Q}_p)$.

4. Let $G = \mathrm{GL}(2, \mathcal{K}) \supset \mathrm{GL}(2, \mathcal{O}) =: K$. Let $B = NH \subset G$ be the Borel subgroup of upper triangular matrices. Prove that

a) $G = B \cdot K = N \cdot H \cdot K$ (Iwasawa decomposition).

b) $\displaystyle\int_G f(g)dg = \int_N \int_H \int_K \left\| \frac{a_1}{a_2} \right\|^{-1} f(nak)\, dn\, da\, dk$ for $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$.

5. Prove that $\displaystyle\int_G f(g)dg = \int_N \int_H \int_N \left\| \frac{a_1}{a_2} \right\|^{-1} f(naw_0u)\, dn\, da\, du$ for $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.