CHARACTERS, IRREDUCIBLES, FOURIER TRANSFORM.

Problem 1. Let $\mathbb{C}^n$ be the standard representation of S_n and $\mathbb{C}^n = V \oplus \mathbb{C}_{triv}$ is its decomposition into irreducible representations. Prove that $\Lambda^k V$ is irreducible for any $1 \leq k \leq n$.

Problem 2. Let T be a faithful representation of G and let χ_T be the corresponding character.
а) Show that $|\chi(g)| \leq \dim T$ and $\chi(g) = \dim T$ only for $g = e$.
б) Compute the character χ of $(T \oplus \mathbb{C}_{triv})^{\otimes N}$ and $\langle \chi_T, \chi \rangle$.
в) Show that any irreducible occurs as an irreducible summand for $T^{\otimes N}$ for some N.

Definition. An algebraic integer is an element $z \in \mathbb{C}$ which is a root of a monic polynomial with integer coefficients.

Proposition. Algebraic integers form a subring of $\mathbb{C}$.

Problem 3. а) Let $C \subset G$ be a conjugacy class. Show that the element $\varphi_C = \sum_{g \in C} g$ is central in $\mathbb{C}[G]$ and hence acts by a scalar λ_C on any irreducible representation.
б) Show that the ring $\mathbb{Z}[G]$ is Noetherian $\mathbb{Z}$-module. Deduce from this fact that λ_C is an algebraic integer.
в) Let T be an irreducible representation of finite group G. Compute $\langle \chi_T, \chi_T \rangle$ and show that $\dim T \mid |G|$.

Problem 4. Show that characters of a finite abelian group form a group $\hat{G}$ which is isomorphic to G (not canonically).

Definition. Let $f : G \to \mathbb{C}$ be a function, G is a finite abelian group. The Fourier transform of f is a function $\hat{f} : \hat{G} \to \mathbb{C}$, $\hat{f}(\chi) = \sum_{x \in G} f(x)\chi(x)$. The Fourier transform is a map

$$\mathcal{F}_G = \mathcal{F} : Fun(G) \to Fun(\hat{G})$$

Problem 5. ("Fourier inversion formula") Let $S_G : Fun(G) \to Fun(G), f \mapsto g$ such that $g(x) = f(x^{-1}) \,\forall\, x \in G$. Show that $\mathcal{F}_{\hat{G}} \circ \mathcal{F}_G = S_G$. Equivalently, for any $f \in Fun(G)$

$$f(x) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(x)\overline{\chi(x)}.$$

Let G be an arbitrary finite group. Then there is an isomorphism of vector spaces $\alpha : \mathbb{C}[G] \to Fun(G)$, $\alpha(x) = \delta_x$ for any $x \in G$. Define the convolution on $Fun(G)$:

$$f_1 * f_2 = \alpha(\alpha^{-1}(f_1)\alpha^{-1}(f_2)).$$

Notation: $Fun_*(G) = (Fun(G), *)$.
Problem 6. а) Show that $(f_1 * f_2)(x) = \sum_{yz=x} f_1(y)f_2(z) = \sum_{y \in G} f_1(y)f_2(y^{-1}x)$
б) Let G be a finite abelian group. Then $\mathcal{F} : Fun_*(G) \to Fun(\hat{G})$ is an isomorphism of algebras.
в) ("Plancherel"theorem") $\mathcal{F}_G$ is a unitary isomorphism (here we use the second orthogonality formula for characters in order to define a Hermitian form on $Fun(\hat{G})$).