Task 1: holomorphic functions, Cauchy formula, Taylor series. Deadline: February 13, 2024

January 26, 2024

Problem 1. Let $U \subset \mathbb{C}^n$ be a domain, and let $\Gamma \subset U$ be a regular holomorphic curve: a one-dimensional complex submanifold in U. Let α be a closed path in Γ that is contractible in Γ (i.e., contractible as a closed path in Γ). Consider a *holomorphic 1-form,* i.e., a 1-form $\omega = \sum_{j=1}^n f_j(z)dz_j$ where $f_j(z)$ are holomorphic functions on U.

a) Prove that the integral along α of the form ω vanishes.

b) Is it true that for every holomorphic 1-form ω and *every* closed path α in U the integral of the form ω along α always vanishes?

Problem 2. Find convergence domain for the Taylor series at the origin of the following functions:

a) $\ln(1 + z_1 - 2z_2^2)$; b) $\frac{1}{1-(z_1-z_2)^2+z_3^2}$.

Problem 3. Prove that the domain of convergence of any Taylor series is always *logarithmically convex:* if two points z, w are contained in the convergence domain, then for every $\alpha \in [0, 1]$ the closed polydisk $\overline{\Delta_{R(\alpha)}}$, $R_j(\alpha) := |z_j|^\alpha |w_j|^{1-\alpha}$, is also contained in the convergence domain.

The **Liouville Theorem** on functions of one complex variable states that *a function holomorphic and bounded on all of $\mathbb{C}$ is constant.*

Prove the following extensions of the Liouville Theorem to two variables.

Problem 4. Prove that every bounded function holomorphic on $\mathbb{C}^2 \setminus K$ is constant, where

a) K is a ball;

b) K is a complex line;

d)* $K = \mathbb{R}^2 \subset \mathbb{C}^2$ is the real plane.

Problem 5. Prove that every function holomorphic on the complement $\Delta_{(1,1)} \setminus S \subset \mathbb{C}^2$ extends holomorphically to all of $\Delta_{(1,1)}$, where

a) $S = \{\frac{1}{2} < |z_1| < 1\} \times \{0\}$;

b) $S = \mathbb{R}^2 \setminus \{|z_1|^2 + |z_2|^2 < \frac{1}{2}\}$;

c)* $S = \mathbb{R}^2$.

Hint to c). Consider the fibration of the space $\mathbb{C}^2$ by parabolas $iz_2 = z_1^2 + \varepsilon$. Try to adapt the proof of two-dimensional Hartogs Theorem (with argument on fibration by parallel lines) to this parabolic fibration.