Seminars 1–3: holomorphic functions, Cauchy formula, Taylor series, Hartogs separate holomorphicity theorem

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Problem 1. Find the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of the following $\mathbb{R}$- linear operators $L : \mathbb{C}^2 \to \mathbb{C}$, here $z = (z_1, z_2)$, $z_k = x_k + iy_k$:

a) $L(z) = x_1 + y_1$;

b) $L(z) = x_1 + y_2$;

c) $L(z) = x_1 + 2iy_2$;

d) Homework: $(1 + i)x_1 + iy_1 + 2x_2 + 3y_2$.

Problem 2. Are the following functions of two variables $f(z) = f(z_1, z_2)$ holomorphic at the origin?

a) $f(z) = x_1 + iy_2$;

b) $f(z) = x_1^2 + 2ix_1y_1 + y_1^2$;

c) $f(z) = x_1^2 + 2ix_1y_1 - y_1^2$;

d) $f(z) = \frac{z_1+z_2}{1+z_1}$;

e) Homework $f(z) = \frac{z_1^2+z_2^3}{z_1^2+z_2^2}$;

f) Homework $f(z) = \frac{z_1^4+z_2^4}{z_1^2+z_2^2}$.

Problem 3. Find which ones of the above functions are

a) continuous in a neighborhood of zero;

b) separately holomorphic (that is, holomorphic in each individual variable z_k) in a neighborhood of the origin.

Problem 4. Calculate the following integrals:

a) $\oint_{\{|z|=1\}} \frac{\sin\zeta+1}{\zeta} d\zeta$, $z \in \mathbb{C}$;

b) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2}{\zeta_1-\frac{1}{2}} d\zeta_1 d\zeta_2$;

c) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2+1}{\zeta_1(\zeta_2-\frac{1}{2})}$;

d) Homework $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=\frac{1}{3}\}} \frac{\cos\zeta_1+\zeta_2}{\zeta_1(\zeta_2-\frac{1}{2})}$.

Problem 5. Write the Taylor series for the following functions at the origin (power series in the usual lexicographic order). Find their convergence domains.

a) $f(z) = \frac{1}{1-z_1z_2^2}$;

b) $f(z) = \frac{1}{1-z_1-z_2^2}$;

c) $f(z) = \frac{1}{(1-z_1)(1+z_2)}$;

d) Homework $f(z) = \frac{1}{(1-(z_1+z_2)^2)(1-z_2)}$;

e) Homework $f(z) = \sin(z_1 + z_2^2)$.

Problem 6. Find the convergence domains of the power series obtained from the series below by opening the brackets and putting monomials in lexicographic order:

a) $\sum_{k=1}^{\infty} k(z_1^2 + 4z_2^2)^k$.

b) $\sum_{k=0}^{\infty} 2^{-k}(z_1 z_2 + z_3^3)^k$.

The Taylor series at the origin of the functions:

c) $\frac{\sqrt{1+(z_1+2z_2)^k}}{1+z_1}$.

d) $\ln(1 + z_1 + z_2 z_3)\sqrt{1 + z_1 z_2}$.

Problem 7. Find the Taylor series at the origin of the functions

a) $((1 - z_1)(1 - z_2)\dots(1 - z_n))^{-1}$.

b) $((1 - z_1)(1 - 2z_2)\dots(1 - nz_n))^{-1}$.

c) $\ln(1 - z_1)\dots\ln(1 - z_n)$.

d) $\exp(z_1 + \dots + z_n)$.

Problem 8. Find the partial derivatives of the above functions a), b), c) at the origin.

Problem 9. Find the function whose Taylor series at the origin is $\sum_{k,n\geq 1} knz_1^k z_2^n$.

Problem 10. Prove that every bounded function holomorphic on $\mathbb{C}^2 \setminus \{(0, 0)\}$ is constant.

Problem 11. Prove the following multidimensional analogue of Hartogs' erasing singularity theorem. Let $R = (R_1, \dots, R_n)$, $R_j > 0$, $1 \leq k < n$, $r = (r_1, \dots, r_k)$, $r_s < R_s$. Set $R^k = (R_1, \dots, R_k)$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. Let $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ be an open subset. Let $z = (z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$,

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}},\ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n,\ \Omega = A \cup B.$$

Then every function holomorphic on Ω extends holomorphically to the whole polydisk $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.

Problem 12. Prove that every function holomorphic on the complement $\Delta_{(1,1)} \setminus S \subset \mathbb{C}^2$ extends holomorphically to all of $\Delta_{(1,1)}$, where

a) $S = \{\frac{1}{2} < |z_1| < 1\} \times \{0\}$;

b) $S = \mathbb{R}^2 \setminus \{|z_1|^2 + |z_2|^2 < \frac{1}{2}\}$;

c)* $S = \mathbb{R}^2$.

Hint to c). Consider the fibration of the space $\mathbb{C}^2$ by parabolas $iz_2 = z_1^2 + \varepsilon$. Try to adapt the proof of two-dimensional Hartogs Theorem (with argument on fibration by parallel lines) to this parabolic fibration.