Seminar 4: analytic sets, Weierstrass polynomials.

February 6, 2024

Problem 1. Show that if M is a connected complex manifold and $A \subset M$ is a nowhere dense analytic subset, then $M \setminus A$ is path connected.

Problem 2. Let $U \subset \mathbb{C}^n$ be an open domain, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let its zero locus $Z := \{f = 0\}$ be non-empty. Let $Z_{crit} \subset Z$ denote the subset consisting of critical points of the function f with critical value 0. Let the complement $Z \setminus Z_{crit}$ be dense in Z. Show that each point $p \in Z_{crit}$ is a singular point of the zero locus Z: this means that there exists no neighborhood $V = V(p) \subset U$ such that $Z \cap V$ is a submanifold in V.

Hint. Suppose to the contrary that Z_f is a local submanifold at a point $x \in Z_f^s$. Then there exists a neighborhood $W = W(x)$ and a biholomorphism H that sends W to a domain $V \subset \mathbb{C}^n$ and sends $Z_f \cap W$ to a coordinate subspace of codimension 1, say, $z_n = 0$. Then the line L through $H(x)$ parallel to the z_n-axis intersects $H(W)$ once, and so does any close line. But one can find a parallel line L' arbitrary close to L that intersects $H(W)$ at least twice: the restriction to L of the function $f \circ H^{-1}$ has zero $H(x)$ of total multiplicity bigger than one, since its differential at $H(x)$ is zero.

Problem 3. (Homework) Consider an affine chart $\mathbb{C}^n$ in complex projective space $\mathbb{CP}^n$. Prove that the closure in $\mathbb{CP}^n$ of the zero set in $\mathbb{C}^n$ of arbitrary polynomial is an analytic subset in $\mathbb{CP}^n$. Deduce that the union of the infinity hyperplane and the zero set is an analytic subset in $\mathbb{CP}^n$.

Problem 4.

a) Show that the subset $\{w + e^{\frac{1}{z}} = 0\} \subset \mathbb{C}^2 \setminus \{z = 0\}$ is an analytic subset, but its closure in $\mathbb{C}^2$ is not an analytic subset in $\mathbb{C}^2$.

b) Find the minimal analytic subset in $\mathbb{C}^2$ that contains the latter closure.

Problem 5. Prove **Weierstrass Division Theorem.** *Let $g(z_1, w) = z_1^d + a_1(w)z_1^{d-1} + \cdots + a_d(z_1)$ be a Weierstrass polynomial of degree d in $z_1 \in \mathbb{C}$; here $w \in \mathbb{C}^{n-1}$ (more precisely, we deal with its germ at $0 \in \mathbb{C}^n$): $a_1(0) = \cdots = a_d(0) = 0$. For every germ of holomorphic function $f(z_1, w)$ there exist a unique germ of holomorphic function $h(z_1, w)$ and a unique polynomial $p(z_1, w) = a_0(w)z_1^{\nu} + \cdots + a_{\nu}(w)$ in z_1, $\nu = deg_{z_1} p < d$, with coefficients $a_j(w)$ holomorphic at $w = 0$ such that*

$$f = hg + p. \tag{1}$$

a) Prove a version for one variable: for every holomorphic function $F(u)$ of one variable u on a closed disk $\overline{D_\delta}$, every $d \in \mathbb{N}$ and every polynomial $g(u)$ of degree d with all the roots contained in D_δ there exist a unique holomorphic function h on D_δ and a unique polynomial $p(u)$ of degree less than d such that

$$F(u) = h(u)g(u) + p(u).$$

b)* Let D_δ and a polydisk $\Delta \subset \mathbb{C}_w^{n-1}$ centered at the origin be such that for every $w \in \Delta$ all the zeros of the polynomial $g(z_1, w)$ in z_1 lie in D_δ, and f is holomorphic on $\overline{D}_\delta \times \Delta$. Prove formula (1) for

$$h(z_1, w) = \frac{1}{2\pi i} \oint_{|u|=\delta} \frac{f(u, w)}{g(u, w)} \frac{du}{u - z_1}.$$