Seminars 6-7. Analytic sets, Weierstrass polynomials, irreducibility

February 20, 2024

Problem 1. Finish proof of Weierstrass Division Theorem.

Hint. The hint from the previous seminar task was correct: the integral is indeed equal to g, not to p, by correctly calculating the residues.

NB. A student Arseniy Kazankov was going to present another proof.

Problem 2. Using Remmert Proper Mapping Theorem prove that the image of an irreducible analytic set under a proper holomorphic mapping is an *irreducible* analytic set. That is, consider an irreducible analytic subset $A \subset M$ in a complex manifold M and a holomorphic mapping $f : M \to N$ to a complex manifold N with proper restriction $f|_A : A \to N$. Then $f(A) \subset N$ is an irreducible analytic subset.

Problem 3. Discuss the notion of *germ* of analytic subset. Define irreducible germs.

a) State and prove sufficient irreducibility condition for germs in terms of connectivity of regular part.

b) State and prove the statement from the previous problem for germs;

Problem 4. Find a polynomial having irreducible germ at $(0, 0)$ that vanish on the parametrized curve $t \mapsto (t^2, t^4 + t^5)$.

Remark. This is an example of bijectively parametrized curve of type $(t^q, ct^p(1 + o(1))$, $1 < q < p$, with q, p being *not coprime,* and a *singular quadratic germ.*

Problem 5. Prove that a one-dimensional analytic subset in $\mathbb{CP}^2$ is holomorphically parametrized by a compact (may be disconnected) Riemann surface such that the parametrization is injective outside at most a finite number of points.

Problem 6. Check and prove whether the following germs of functions at zero are irreducible.

a) $f(z, w) = z^p + w + O(|z|^2)$;

b) $z_1^p - z_2^q$, the answer depends on the choice of the pair (p, q);

c) $f(z, w) = w^2 + z^2 + w^3 z^4$;

d) $f(z, w) = z^3 + w^3 + z^4 + w^5$;

e) $f(z, w) = z^3 + w^2 + w^3$.

f)* $w^p + z_1^2 + z_2^2$, $p \geq 2$.

g)* $w^p + z_1^2 + z_2^3$, $p \geq 2$.

Problem 7. For this or the next time, one can also present proof of Noetherianity of local ring, step by step, leaving proofs of some of the steps for students.

Problem 8. Define the Newton diagram of a germ of holomorphic function. State the theorem saying that if the germ is irreducible, then the Newton diagram consists of one edge. Discuss counterexamples in the other direction. Discuss (exercise?) the statement that if the Newton diagram consists of one edge connecting the points $(p, 0)$, $(0, q)$, and p, q, are coprime, then the germ is irreducible. Present some exercises on Newton diagrams.