Task 2: analytic sets, Weierstrass polynomials. Newton diagrams. Deadline: April 4

Let us recall the following material.

Theorem. *A germ of function $f(z, w) \in \mathcal{O}_2$, is irreducible, if and only if its zero locus is a germ of holomorphically bijectively parametrized curve and $df(x) \not\equiv 0$ on its zero locus.* Equivalently, *a germ f with $f(0, w) \not\equiv 0$ is irreducible, if and only if its zero locus can be parametrized by $t \mapsto (t^q, ct^p(1 + o(1)))$, $q, p \in \mathbb{N}$, and for every small z the function $f(z, w)$ in variable w has simple roots converging to zero, as $z \to 0$.*

Given a germ of function $f(z, w) \in \mathcal{O}_2$, consider the convex hull of the set of quadrants $(m, n) + \mathbb{R}^2_+$ through all the bidegrees (m, n) of Taylor monomials of the function f. It is a polygon with finite number of edges. The union $\mathcal{N}_f$ of those edges that do not lie in the coordinate axes is called the *Newton diagram.*

Theorem. *1) If a germ $f(z, w) \in \mathcal{O}_2$ is irreducible, then $\mathcal{N}_f$ consists of just one edge.*
2) If f is reducible, then for each its irreducible factor the (unique) edge of its Newton diagram is parallel to some edge in $\mathcal{N}_f$, and each edge in $\mathcal{N}_f$ is realized by at least one factor.

Problem 1. Let $A \subset \mathbb{CP}^2$ be an analytic subset that is locally defined (in a neighborhood of each point) as zero locus of just one holomorphic function in this neighborhood. Prove that

a) the number of its singular points is finite;

b) the set A is holomorphically parametrized by a compact (may be disconnected) Riemann surface so that the parametrization is injective outside at most a finite number of points.

Problem 2. Let $f \in \mathcal{O}_2$ be an irreducible germ. Consider the local chart (z, w) with the z-axis being tangent to its zero locus at the origin (which is a parametrized curve). Such a coordinate system is called *adapted* to the parametrized curve.

Prove that in the adapted chart the (unique) edge of the Newton diagram has angle strictly less than $\frac{\pi}{4}$ with the horizontal axis.

Problem 3. Show that a) if $f \in \mathcal{O}_2$ is an irreducible germ, then its lower homogeneous part is a power $(az + bw)^d$; b)* in general the number of irreducible factors of f is no less than the number of zero lines of its lower homogeneous part.

Problem 4. Prove the Morse Lemma for holomorphic functions in two variables. *Let a function $f(z, w) \in \mathcal{O}_2$ vanish at the origin together with its first partial derivatives and have the form $f(z, w) = Q(z, w)+$* terms of order ≥ 3, *$Q(z, w)$ being a non-degenerate quadratic form. Then $f(z, w)$ can be brought to $f(z, w) = z^2 + w^2$ by a germ of biholomorphic coordinate change.*

Problem 5. Find Newton diagrams of the following Weierstrass polynomials in two variables.

Check (and prove) whether their germs at the origin are irreducible as germs of holomorphic functions. For those that aren't, find the number of the corresponding irreducible factors.

a) $w^3 - z^5$;

b) $w^4 - z^6$;

c) $z_1^p - z_2^q$, the answer depends on the choice of the pair (p, q);

d) $w^4 + z^2w^2 - 2z^4$;

e) $w^4 + 2zw^2 + z^4$;

f) $w^5 + 3zw^3 + 7z^3w + z^5$.