Several Complex Variables. Seminar material for March 19, 2024

Problem 1. Find the lower nonlinear term of k-th iterate of the germ of conformal mapping

$$f(z) = az + z^m + O(z^{m+1}),\ m \geq 2,\ a \in \mathbb{C}^*.$$

Problem 2. Let us construct a polynomial automorphism with an attractive basin being a Fatou–Bieberbach domain. Take polynomial automorphisms

$$f : (z_1, z_2) \mapsto (z_1 + z_2, z_2);\ g : (z_1, z_2) \mapsto (z_1, z_2 + z_1^2).$$

Let us choose a non-resonant diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2),\ \lambda_1 \neq \lambda_2,\ 0 < |\lambda_1|, |\lambda_2| < 1.$$

Set

$$F(z) = \Lambda g \circ f(z) = \begin{pmatrix} \lambda_1(z_1 + z_2) \\ \lambda_2(z_2 + (z_1 + z_2)^2) \end{pmatrix}.$$

Proposition 0.1 *The attractive basin V of the fixed point 0 of the automorphism F is biholomorphically equivalent to $\mathbb{C}^2$. The automorphism F has an additional fixed point $q \neq 0$, hence $V \neq \mathbb{C}^2$ is a Fatou–Bieberbach domain.*

Proof The differential $dF(0)$ has distinct eigenvalues λ_1, λ_2, and hence, is conjugated to the diagonal matrix. Therefore, F is linearizable on V, by Linearization Theorem and non-resonance condition. The system of equations on fixed points has the form

$$\begin{cases} z_1 = \lambda_1(z_1 + z_2) \\ z_2 = \lambda_2(z_2 + (z_1 + z_2)^2) \end{cases} \tag{0.1}$$

The first equation of the system is equivalent to each one of the two following equations:

$$z_1 = \frac{\lambda_1 z_2}{1 - \lambda_1},\ z_1 + z_2 = z_2(1 + \frac{\lambda_1}{1 - \lambda_1}) = \frac{z_2}{1 - \lambda_1}.$$

Substituting the latter expression for $z_1 + z_2$ to the second equation in (0.1) and dividing it by z_2 yields

$$1 + \frac{z_2}{(1 - \lambda_1)^2} = \lambda_2^{-1}.$$

This yield a solution

$$z_2 = (1-\lambda_1)^2(\lambda_2^{-1}-1),\ z_1 = \frac{\lambda_1 z_2}{1-\lambda_1} = \frac{\lambda_1}{\lambda_2}(1-\lambda_1)(1-\lambda_2)$$

of system (0.1), and hence, an additional fixed point of the mapping F. The proposition is proved. □

Problem 3. Show that intersection of attracting basin (Fatou-Bieberbach domain) with a line is simply connected (Maximum Principle). Deduce that unit disk is embedded as a submanifold in $\mathbb{C}^2$.